Proof that the results of regular Sturm-Liouville problem hold in our work

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Contents

1 Notations

We introduce some necessary notations from the chapter *A Catalogue of Sturm-Liouville Differential Equations* by W. Norrie Everitt [1] in Page 275 of the book *Sturm-Liouville Theory: Past and Present* [2].

Lebesgue integration space of complex-valued functions defined on the interval I is denoted as $L^1(I)$. The **local integration space** $L^1_{loc}(I)$ is the set of all complex-valued functions on I which are Lebesgue integrable on all compact sub-intervals $[a,b] \subseteq I$.

Absolute continuity, with respect to Lebesgue measure, is denoted by AC; the space of all complex-valued functions defined on I which are absolutely continuous on all compact sub-intervals of I, is denoted by $AC_{loc}(I)$.

2 Boundary Value Problem

The governing equations of the Euler buckling problem for one half column is

$$(\tilde{I}\tilde{v}')' + \beta\tilde{v} = 0, \quad \forall \eta \in (0,1), \tag{1a}$$

$$\tilde{I}\tilde{v}'\big|_{\eta=0} = 0, \tag{1b}$$

$$\tilde{v}\big|_{\eta=1} = 0. \tag{1c}$$

where $\tilde{I}(\eta)$ is the non-dimensional second moment of inertia of the column, $\tilde{v} = \tilde{w}'$ is the non-dimensional rotation angle of the column. Due to the symmetry of the problem, we only consider the half of the column. At the end $\eta = 0$, the beam is hinged so the bending moment $\tilde{I}\tilde{v}'$ vanishes. The end $\eta = 1$ is the middle point, *i.e.* the symmetry point, of the original beam. The coefficient function $\tilde{I}(\eta)$ is non-zero at $\eta = 1$ and may vanish at $\eta = 0$.

If we change the variables $\tilde{I}(\eta)$, $\tilde{v}(\eta)$, η, β to $p(x)$, $y(x)$, x, λ, respectively, the above problem is equivalent to the abstract math problem as follows, {eq:AbstractBVP}

$$-\big(p(x)y'(x)\big)' = \lambda y(x), \quad \forall x \in (0,1), \tag{2a}$$ {eq:GE}

$$p(0)y'(0) = 0, \tag{2b}$$ {eq:bc1}

$$y(1) = 0. \tag{2c}$$ {eq:bc2}

The coefficient function $p(x)$ is continuous in $[0,1]$, $p(x) > 0$ for $0 < x \le 1$ and $p(0) \ge 0$.

This is a special case of the general Sturm-Liouville differential equation

$$M[y](x) \equiv -\big(p(x)y'(x)\big)' + q(x)y(x) = \lambda w(x)y(x), \quad \forall x \in (a,b), \tag{3}$$ {eq:generalSL}

where $\lambda \in \mathbb{C}$ is a complex-valued spectral parameter. The set of Sturm-Liouville coefficients $\{p,q,w\}$ has to satisfy the minimal conditions

(1) $p,\ q,\ w\ : (a,b) \to \mathbb{R}$,

(2) $p^{-1},\ q,\ w \in L^1_{loc}(a,b)$,

(3) w is a weight function on (a,b), which means that $w(x) : (a,b) \to \mathbb{R}$ is a Lebesgue measurable function and $w(x) > 0$ for almost all $x \in (a,b)$.

For the problem (2), all of the above three minimal conditions are satisfied.

3 Endpoints classification

We introduce the endpoints classification from the chapter *A Catalogue of Sturm-Liouville Differential Equations* by W. Norrie Everitt [1] in Page 277 of the book *Sturm-Liouville Theory: Past and Present* [2].

Suppose given the interval (a,b) and the set of coefficients $\{p,q,w\}$.

(1) The endpoint a is **regular** if

- (I) $a > -\infty$, and
- (II) $p^{-1},\ q,\ w \in L^1(a,c],\ \forall\, c \in (a,b)$.

(2) The endpoint a is **singular** if it is not regular, *i.e.*,

- (I) either $a = -\infty$,
- (II) or $a > -\infty$ but $\int_a^c [|p(x)|^{-1} + |q(x)| + |w(x)|]dx = +\infty,\ \forall\, c \in (a,b)$.

If a is a singular endpoint, there are two classification subcases as follows:

(1) It is **limit-point** if for some $\lambda \in \mathbb{C}$, at least one solution $y(\cdot,\lambda)$ of the differential equation (2) satisfies

$$\int_a^c w(x)|y(x,\lambda)|^2 dx = +\infty,\ \forall\, c \in (a,b). \tag{4}$$ {eq:LP}

(2) It is **limit-circle** if for some $\lambda \in \mathbb{C}$, all solutions $y(\cdot,\lambda)$ of the differential equation (2) satisfy

$$\int_a^c w(x)|y(x,\lambda)|^2 dx < +\infty,\ \forall\, c \in (a,b). \tag{5}$$ {eq:LC}

Remarks:

(1) We stress the point made above that although the spectral parameter λ is involved in the endpoint classification, it can be shown that this classification is independent of λ and depends only on the interval (a,b) and the set of coefficients $\{p,q,w\}$.

(2) All the above remarks apply equally well, with change of notation, to the classification cases of endpoint b; note that the classification of a and of b are independent of each other.

According to above classification, for our problem (2), $x = 1$ is a regular endpoint. The singularity of the endpoint $x = 0$ depends on the behavior of $1/p(x)$ as $x \to 0$. We do series expansion of $p(x)$ around $x = 0$ and take the leading order term x^α. We have the following discussion:

(1) If $\alpha \le 0$, which means $p(0) \ne 0$, the endpoint $x = 0$ is **regular**.

(2) If $0 < \alpha < 1$, which means $p(0) = 0$ but $\int_0^c 1/p(t)dt < +\infty,\ \forall\, c \in (0,1)$, the endpoint $x = 0$ is still **regular**.

(3) If $\alpha \ge 1$, which means $p(0) = 0$ and $\int_0^c 1/p(t)dt = +\infty,\ \forall\, c \in (0,1)$, the endpoint $x = 0$ is **singular**.

Therefore, if the singularity of $1/p(x)$ is not bad ($0 < \alpha < 1$), the problem (2) is regular and all the results of regular Sturm-Liouville theory hold. (Please see Ch. 11 of *Elementary differential equations and boundary value problems* by Boyce and Di Prima [3] for results of regular Sturm-Liouville theory.) However, if the singularity of $1/p(x)$ is large ($\alpha \geq 1$), problem (2) becomes a singular Strum-Liouville problem. For example, for columns with **ellipse** or **Clausen** profile, the problem is singular. We are going to explore whether the following results hold in our problem:

(1) The self-adjoint relation holds.

(2) The problem consists only a discrete set of eigenvalues.

(3) All the eigenvalues are real.

(4) The corresponding eigenfunctions form a complete, orthogonal set in the Hilbert function space.

(5) The expansion of a given continuous function f in terms of a series of eigenfunctions is convergent.

Taking $\lambda = 0$, the differential equation (2a) becomes $-(p(x)y'(x))' = 0$. Approximating $p(x)$ by its leading order term x^α, $\alpha \geq 1$, we get the non-trivial solution of $y(x)$ as

$$y(x) \sim C_1 \frac{x^{1-\alpha}}{1-\alpha} + C_2, \tag{6}$$

where C_1 and C_2 are integration constants. We can estimate the L^2 norm of $y(x)$. For $c \in (0,1)$

$$\int_0^c |y(x)|^2 dx \sim \lim_{x \to 0} \left[\frac{C_1^2}{(1-\alpha)^2(3-2\alpha)} x^{3-2\alpha} + \frac{C_1 C_2}{(1-\alpha)(2-\alpha)} x^{2-\alpha} + C_2^2 x \right]. \tag{7}$$

For $\alpha < \frac{3}{2}$, $\int_0^c |y(x)|^2 dx < +\infty$, the endpoint $x = 0$ is **limit-circle**; for $\alpha \geq \frac{3}{2}$, $\int_0^c |y(x)|^2 dx = +\infty$, the endpoint $x = 0$ is **limit-point**.

4 Boundary Conditions

For the **regular** endpoint $x = a$, a separated boundary condition should take the form, where $A_1, A_2 \in \mathbb{R}$ with $A_1^2 + A_2^2 > 0$,

$$A_1 y(a) + A_2 (py')(a) = 0. \tag{8}$$

{eq:reg_bc}

At endpoint $x = 1$, we take $A_1 = 1$ and $A_2 = 0$, which gives the boundary condition (2c). If $x = 0$ is also a **regular** endpoint, then the boundary condition (2b) is just a special form of (8) with $A_1 = 0$ and $A_2 = 1$.

For the **singular** endpoint $x = a$, if this is **limit-circle**, then a separated boundary condition should take the form,

$$A_1 [y,u](a) + A_2 [y,v](a) = 0, \tag{9}$$

{eq:LC_bc}

where

(1) $A_1, A_2 \in \mathbb{R}$ with $A_1^2 + A_2^2 > 0$,

(2) $u, v : (a,b) \to \mathbb{R}$,

(3) $u, v \in D(T_1) = \{f \in D(M) : f, w^{-1} M[f] \in L^2((a,b);w)\}$,

(4) $[f,g](x) := f(x) p(\bar{g}')(x) - (pf')(x)\bar{g}(x)$,

(5) $[u,v](a) \neq 0$.

Such pair $\{u,v\}$ is always possible; if $\lambda \in \mathbb{R}$, then take $u(\cdot) = u(\cdot,\lambda)$ and $v(\cdot) = v(\cdot,\lambda)$ where $\{u(\cdot,\lambda), v(\cdot,\lambda)\}$ is a real, linearly independent basis of solutions of the differential equation (2a). We can take $u(x) = 1$ and $v(x) = \int_c^x 1/p(t)dt$ where $c \in (0,1]$. They are solutions of (2a) as $\lambda = 0$. At the endpoint $x = 0$, it follows that

$$[u,v](0) = u(0)p(\bar{v}')(0) - (pu')(0)\bar{v}(x) = p(0)(1/p(0)) = 1 \neq 0. \tag{10}$$

We take $A_1 = 1$ and $A_2 = 0$, which gives the boundary condition

$$[y,u](0) = y(0)p(\bar{u}')(0) - (py')(0)\bar{u}(x) = -p(0)y'(0) = 0, \tag{11}$$

which is same as the boundary condition (2b).

If the endpoint $x = 0$ is **limit-point**, no separated boundary condition at $x = 0$ is required nor allowed. In fact, there is a connection between the endpoint classification of a and the limit $[f,g](a)$:

$$a \text{ is limit-point if and only if } [f,g](a) = 0,\ \forall f,g \in D(T_1). \tag{12}$$

In summary, in our problem (2), the boundary conditions (2b) and (2c) are valid for all classification of endpoints $x = 0$ and $x = 1$.

5 Spectrum Properties

Please see section 8 in the paper *Algorithm 810: The SLEIGN2 Sturm-Liouville Code* by Bailey *et al.* [4] for reference.

The endpoint $x = 1$ is **regular**. The spectrum property of the problem (2) depends on the classification of endpoint $x = 0$.

(1) If $x = 0$ is **regular** or **limit-circle**, then:

- (I) The spectrum is always *discrete*, *simple* and *bounded below*.
- (II) The eigenvalues are indexed as $\{\lambda_n : n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}\}$, with $\lambda_n < \lambda_{n+1}$ for all $n \in \mathbb{N}_0$ and $\lim_{n\to\infty} \lambda_n = +\infty$.

(2) If $x = 0$ is **limit-point**, then:

- (I) The spectrum is always *simple* but may or may not be *discrete*, and may or may not be *bounded below*.
- (II) If the spectrum is discrete and bounded below then the eigenvalues are indexed as $\{\lambda_n : n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}\}$, with $\lambda_n < \lambda_{n+1}$ for all $n \in \mathbb{N}_0$ and $\lim_{n\to\infty} \lambda_n = +\infty$; the nth eigenfunction has exactly n zeros in the open interval $(0, 1)$.
- (III) If the continuous (essential) spectrum is bounded below, say by σ, then:
  - (A) There may be no eigenvalues below.
  - (B) There may be a finite number of eigenvalues below σ, indexed as $\{\lambda_n : n \in \{0, 1, 2, \ldots, N\}\}$ with $N \geq 0$, and $\lambda_n < \lambda_{n+1} \leq \sigma$ for $n = 0, 1, \ldots, N-1$; every eigenfunction belonging to n has exactly n zeros in the open interval $(0, 1)$.
  - (C) There may be a countable infinity of eigenvalues below σ, indexed as $\{\lambda_n : n \in \mathbb{N}_0\}$ with $\lambda_n < \lambda_{n+1} < \sigma$ for $\lim_{n\to\infty} \lambda_n = \sigma$; every eigenfunction belonging to n has exactly n zeros in the open interval $(0, 1)$.
- (IV) There may be a countable infinity of eigenvalues below σ such that for which the spectrum is discrete but unbounded above and below, say $\{\lambda_n : n \in \mathbb{Z}_0 = \{\ldots, -2, -1, 0, 1, 2, \ldots\}\}$ with $\lim_{n\to\pm\infty} \lambda_n = \pm\infty$; in such cases all eigenfunctions have infinitely many zeros.

Thus, to guarantee that our buckling problem and perturbation analysis to be valid, the endpoint $x = 0$ must be **regular** or **limit-circle**. That is, for $p(x) \sim x^{\alpha}$, $\alpha < \frac{3}{2}$ must be satisfied. This condition is sufficient but not necessary.

For columns with constant ($\alpha = 0$) and Clausen profile ($\alpha = 4/3$), the endpoint $x = 0$ is regular and limit-circle, respectively. For columns with ellipse profile ($\alpha = 2$), the endpoint $x = 0$ is limit-point. Therefore, the perturbation analysis may not apply for ellipse columns.

reference

[1] W. Norrie Everitt. A catalogue of sturm-liouville differential equations. In *Sturm-Liouville Theory*, pages 271–331. Birkhäuser Basel, 2005.

[2] Werner O. Amrein, Andreas M. Hinz, and David P. Pearson, editors. *Sturm-Liouville theory: past and present*. Birkhäuser, Basel ; Boston, 2005.

[3] William E. Boyce and Richard C. DiPrima. *Elementary differential equations and boundary value problems*. Wiley, Hoboken, NJ, tenth edition edition, 2012.

[4] P. B. Bailey, W. N. Everitt, and A. Zettl. Algorithm 810: The sleign2 sturm-liouville code. *ACM Trans. Math. Softw.*, 27(2):143–192, June 2001.