$$I[u,w] = \frac{E}{2}\int_0^L \Big[\underbrace{A(x)\left(u_{,x} + \frac{w_{,x}^2}{2}\right)^2}_{\approx 0} + I\, w_{,xx}^2\Big] dx - \int_{x=0}^{L} f(x)\, u(x)\, dx \quad -(1)$$ 1:30 pm

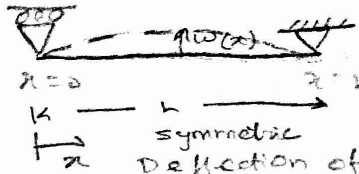
⇒ Inextensibility condition $u_{,x} = -\frac{w_{,x}^2}{2}$ -(2)

$$I[u,w] = \frac{E}{2}\int_0^L I(x)\, w_{,xx}^2\, dx - \int_{x=0}^{L} f(x) u(x)\, dx \quad (1a)$$

⇒ The forcing function $f(x) = 2\pi\sigma_{11}\left(\frac{a_2}{a_1}\right)^2\left(\frac{L}{2} - x\right)$

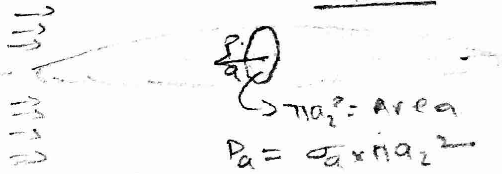
$f(x) = 2 \times \frac{P_a}{L}\left(\frac{1}{2} - \frac{x}{L}\right)$

$P_a = \sigma_a \pi a_2^2$ $\quad \sigma_a$: (1b)



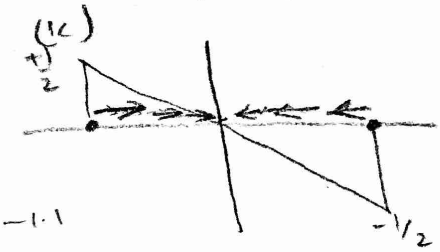
symmetric Deflection of the Beam-column.

$$f(x) = \left(\frac{2P_a}{L}\right)\left(\frac{1}{2} - \frac{x}{L}\right) \quad -(3)$$



INTEGRATE BY parts

$$I[u,w] = \frac{1}{2}E\int_0^L \underbrace{I w_{,xx}^2\, dx}_{1} - \underbrace{F(x)u(x)\Big|_0^L}_{2} + \int_0^L \underbrace{F(x) u'(x)\, dx}_{3} ; \quad -1.1$$



where $F(x) = \int_0^x f(y)\, dy$

$F(x) = P_a\left(\frac{x}{L} - \left(\frac{x}{L}\right)^2\right)$ -4

we need to assume that $f(y)$ has no singularity at $x=0$.. Then I believe that 2 can be dropped without problem

so dropping ② and making use of the inextensibility condition. $u' = -\frac{w_{,x}^2}{2}$, we get that

$\frac{EI w_{,\xi\xi}^2 x^2}{L^4 L^2}$

$$\Pi[w] = \frac{1}{2}\int_0^L \left(EI w_{,xx}^2 - F(x)\, w_{,x}^2\right) dx ;$$

$$\mathcal{V} = \{ w \in H^2([0,L]) : w(0) = 0,\ w(L) = 0 \}$$

Non-dimensionalization: $\xi = x/L$; $\hat{I}(\xi) = I(x)/L^4$; $\hat{w}(\xi) = w(x)/L$

$\frac{d}{dx}(\cdot) = \frac{1}{L}\frac{\partial(\cdot)}{\partial\xi}$, $\frac{\partial^2}{\partial x^2}(\cdot) = \frac{\partial^2}{\partial\xi^2}(\cdot)\frac{1}{L^2}$; $\hat{\Pi} = \frac{\Pi}{EL^3}$

Non dimensionalizing the above functional we get that;

$$\hat{\Pi}[\hat{w}] = \frac{1}{2}\int_0^1 \left[\hat{I}(\xi)\hat{w}''^2 - \beta\, \hat{F}(\xi)\hat{w}'^2\right] d\xi ; \quad -1.2$$

where $\beta = \frac{P_a}{EL^2}$, $\hat{F}(\xi) = \xi - \xi^2$;

$$\hat{\mathcal{V}} = \{ \hat{w} \in H^2([0,1]) : \hat{w}(0) = \hat{w}(1) = 0 \}$$

The Strong . FORM

essential conditions

$$\begin{cases} \hat{I}(\xi)\hat{w}''(\xi) = 0 & \text{at } \xi = 0, 1. \quad 3.1 \\ \hat{w}(\xi) = 0 & \text{at } \xi = 0, 1 \quad 3.2 \\ (\hat{I}\hat{w}'')'' + \beta(\hat{F}\hat{w}')' = 0 & \forall \xi \in [0, 1] \quad 3.3 \end{cases}$$

Modification of the Functional using symmetry of Problem

$$\tilde{\pi}[\tilde{\omega}(\cdot)] = \int_0^1 \left(\tilde{I}\tilde{w}''^2 - \beta\tilde{F}\tilde{w}'^2\right) d\eta \quad 1.3$$

$$\tilde{I}(\eta) = 8\hat{I}\left(\tfrac{\eta}{2}\right); \quad \tilde{F}(\eta) = 2\hat{F}(\eta/2)$$

$$\eta = \xi/1/2; \quad \tilde{w}(\eta) = \hat{w}\left(\tfrac{\eta}{2}\right)$$

$$\tilde{\mathcal{V}} = \left\{ \tilde{w} \in H^2([0,1]) : \begin{array}{l} \tilde{w}(0) = 0 \\ \tilde{w}'(1) = 0 \end{array} \right\}$$

(John said that this OK).

Further modificiation of the functional

$$\tilde{\pi}[\tilde{v}(\cdot)] = \int_0^1 \left(\tilde{I}\tilde{v}'^2 - \beta\tilde{F}\tilde{v}^2\right) d\eta \quad - 1.4.$$

$$T\tilde{U} = \{\delta\tilde{v} \in H^1([0,1]) : \delta\tilde{v}(1) = 0\}.$$

$$\tilde{v} = \tilde{w}' \in \tilde{U} = \left\{ \tilde{v} \in H^1([0,1]) : \tilde{v}(1) = 0 \right\}$$

1- Since $\tilde{w} = \int \tilde{v} + c$ and c can be chosen to make $\tilde{w}(0) = 0$. so, I don't see a problem with the above formalism.

strong form:

$$\boxed{(\tilde{I}\tilde{v}')' + \beta\tilde{F}\tilde{v} = 0} \quad \forall \eta \in [0,1] \quad - 3.a.1$$

3-a-2

$$\tilde{I}(0)\tilde{v}'(0) = 0 \quad 3-a.3$$

$$\tilde{v}(1) = 0.$$

(John said that this too is OK)

$$\tilde{I}(\xi;\epsilon) = \tilde{I} + \epsilon \tilde{I}_1(\xi) + O(\epsilon^2) \Rightarrow \tilde{I}(\xi;0) = \tilde{I}$$
$$\tilde{v}(\xi;\epsilon) = \tilde{v}(\xi) + \epsilon \tilde{v}_1(\xi) + O(\epsilon^2) \qquad \tilde{v}(\xi;0) = \tilde{v}$$
$$\tilde{I}(\xi;\epsilon)_{,\xi} = \tilde{I}' + \epsilon \tilde{I}_1' + O(\epsilon^2) \Rightarrow \tilde{I}(\xi;\epsilon)_{,\xi}\big|_{\epsilon=0} = \tilde{I}'$$
$$\tilde{v}(\xi;\epsilon)_{,\xi} = \tilde{v}' + \epsilon \tilde{v}_1'(\xi) + O(\epsilon^2) \Rightarrow \tilde{v}(\xi;\epsilon)_{,\xi}\big|_{\epsilon=0} = \tilde{v}'$$
$$\beta(\epsilon) = \beta + \cancel{\beta_1\epsilon} + \cancel{O(\epsilon^2)}\ \beta_2\epsilon^2 + O(\epsilon^4) \Rightarrow \beta(0) = \beta$$
$$\frac{d\beta}{d\epsilon}\Big|_{\epsilon=0} = 0$$
$$\tilde{v}(\xi;\epsilon)_{,\epsilon} = \tilde{v}_1(\xi) + O(\epsilon) \Rightarrow \tilde{v}(\xi;\epsilon)_{,\epsilon}\big|_{\epsilon=0} = \tilde{v}_1(\xi)$$
$$\tilde{I}(\xi;\epsilon)_{,\epsilon} = \tilde{I}_1(\xi) + O(\epsilon) \Rightarrow \tilde{I}(\xi;\epsilon)_{,\epsilon}\big|_{\epsilon=0} = I_1$$

expansions to perturb soln

The function $\tilde{I}(\xi;\epsilon)$, $\tilde{v}(\xi;\epsilon)$, $\beta(\epsilon)$, should also satisfy 3 a.1 for each ϵ so we get that

$$(\tilde{I}(\xi;\epsilon)\, \tilde{v}(\xi;\epsilon)_{,\xi})_{,\xi} + \beta(\epsilon)\tilde{F}(\xi)\, \tilde{v}(\xi;\epsilon) = 0 \quad \text{for each } \epsilon.$$

So we take a derivative with respect to ϵ and that should to for each ϵ

$$(\tilde{I}(\xi;\epsilon)_{,\epsilon}\, \tilde{v}(\xi;\epsilon)_{,\xi} + \tilde{I}(\xi;\epsilon)\, \tilde{v}(\xi;\epsilon)_{,\xi\epsilon})_{,\xi} + \beta_{,\epsilon}\tilde{F}(\xi)\tilde{v}(\xi;\epsilon) + \beta(\epsilon)\tilde{F}(\xi)\,\tilde{v}(\xi;\epsilon)_{,\epsilon} = 0 \quad \text{for } \epsilon \text{ small enough}$$

~~Thus for $\epsilon=0$~~

$$\tilde{I}(\xi;\epsilon)_{,\epsilon\xi}\tilde{v}(\xi;\epsilon)_{,\xi} + \tilde{I}(\xi;\epsilon)_{,\epsilon}\tilde{v}(\xi;\epsilon)_{,\xi\xi} + \tilde{I}(\xi;\epsilon)_{,\xi}\tilde{v}(\xi;\epsilon)_{,\xi\epsilon} + \tilde{I}(\xi;\epsilon)\tilde{v}(\xi;\epsilon)_{,\xi\xi\epsilon} + \beta_{,\epsilon}\tilde{F}(\xi)\tilde{v}(\xi;\epsilon) + \beta(\epsilon)\tilde{F}(\xi)\tilde{v}(\xi;\epsilon)_{,\epsilon} = 0 \quad \text{for } \epsilon \text{ small enough.}$$

For $\epsilon=0$ we get that

$$\tilde{I}(\xi;\epsilon)_{,\epsilon\xi}\big|_{\epsilon=0}\tilde{v}' + I_1(\xi)\tilde{v}(\xi;\epsilon)_{,\xi\xi}\big|_{\epsilon=0} + \tilde{I}'\tilde{v}(\xi;\epsilon)_{,\xi\epsilon}\big|_{\epsilon=0} + \tilde{I}\tilde{v}(\xi;\epsilon)_{,\xi\xi\epsilon}\big|_{\epsilon=0} + \beta\tilde{F}(\xi)\tilde{v}_1(\xi) = 0 \quad \forall \xi \in [0,1]$$

$$I_1'(\xi)\tilde{v}' + I_1(\xi)\tilde{v}'' + \tilde{I}'\tilde{v}_1' + I(\xi)\tilde{v}_1''(\xi) + \beta\tilde{F}(\xi)\tilde{v}_1(\xi)$$

$$\boxed{(\tilde{I}\tilde{v}_1')' + \beta\tilde{F}\tilde{v}_1 = -\underbrace{(I_1\tilde{v}')'}_{\text{forcing function}}} \quad — (4)$$

~~choose the next proposal and start preparing for that~~

B-C at $\xi=1$

$$\tilde{v}(\xi,\epsilon)\Big|_{\xi=1}=0 \qquad \text{for } \epsilon \text{ small enough}$$

$$\cancel{\tilde{v}(\xi,\epsilon)_{,\epsilon}}$$

$$\tilde{v}(\xi)+\epsilon\tilde{v}_1(\xi)+o(\epsilon^2)\Big|_{\xi=1}=0 \qquad ''$$

$$\tilde{v}(1)+\epsilon\tilde{v}_1(1)+o(\epsilon^2)=0 \qquad ''$$

$$\tilde{v}_1(1)+o(\epsilon^2)=0 \qquad ''$$

$$\tilde{v}_1(1)=0$$

B-C at $\xi=0$.

$$\tilde{v}(\xi;\epsilon)_{,\xi}=\tilde{v}_{,\xi}+\tilde{v}_{1,\xi}\,\epsilon+o(\epsilon^2) \qquad ''$$

$$\tilde{v}(0;\epsilon)_{,\xi}=\tilde{v}_{,\xi}(0)+\tilde{v}_{1,\xi}(0)\,\epsilon+o(\epsilon^2)=0 \qquad ''$$

$$\tilde{v}(0;\epsilon)_{,\xi\epsilon}=\tilde{v}_{1,\xi}(0)+o(\epsilon)=0 \qquad ''$$

$$\tilde{v}(0;0)_{,\xi\epsilon}=\tilde{v}_{1,\xi}(0)=0$$

$$\tilde{v}_1(1)=0$$
$$\tilde{v}_{1,\xi}(0)=0$$

So the Boundary value Problem is.

$$\boxed{(I\tilde{v}_1')'+\beta\tilde{F}\tilde{v}_1=-(I_1\tilde{v}')'} \qquad 4.1$$

$$\tilde{v}_1(1)=0 \qquad 4.2$$

$$\tilde{v}_{1,\xi}(0)=0 \qquad 4.3.$$

This is the exact same boundary ~~conditio~~ value problem as before except there is a forcing term on the R.H.S. In fact β is the eigen value of the Homogenous version of the B.V.P so the $-(I_1\tilde{v}')'$ should be orthogonal to the eigenfunction corresponding to β, which being $\tilde{v}$, thus.

$$\int_{\eta=0}^{1}\tilde{v}(I_1\tilde{v}')'\,d\eta=0 \Rightarrow \tilde{v}(I_1\tilde{v}')\Big|_{\eta=0}^{1}-\int_{\eta=0}^{1}\tilde{v}'I_1\tilde{v}'\,d\eta=0$$

From 3.9.2 and 3.9.3 $\tilde{v}(1)=0$ and $\tilde{v}'(0)=0$. So the sufficient cond. is.

$$\boxed{\int_{\eta=0}^{1}I_1\tilde{v}'^2=0.} \qquad \text{— (5)}$$

Combine this w/ next volume constraint to get I in terms of V

From the constraint that volume is constant we get that

$$(4\pi)^{1/2}\int_{\eta=0}^{1} \tilde I(\eta)^{1/2}\, d\eta = \tilde V \cancel{\sqrt{\tfrac{2}{\pi}}} \Rightarrow (4\pi)^{1/2}\int_{\eta=0}^{1} \tilde I(\eta;\epsilon)^{1/2}\, d\eta = \tilde V \cancel{\sqrt{\tfrac{2}{\pi}}}$$

$$\Rightarrow \int_{\eta=0}^{1} \tfrac{1}{2}\tilde I(\eta;\epsilon)^{-\frac{1}{2}} \frac{d\tilde I}{d\epsilon}(\eta;\epsilon)\Big|_{\epsilon=0} d\eta = \int_{\eta=0}^{1} \tilde I(\eta)^{-\frac{1}{2}} I_1(\eta)\, d\eta = 0 \quad -(6)$$

combing ⑤ + ⑥ we get

$$\int_{\eta=0}^{1} I_1(\eta)\,\tilde I(\eta)^{-\frac{1}{2}} \quad \tilde I(\eta)^{+\frac{1}{2}}\,\tilde v'^2\, d\eta = 0$$

If $\boxed{\tilde I(\eta)^{+1/2}\,\tilde v'^2 = \text{const.}}$ then the above equation holds

∴

$$\tilde I(\eta)^{+\frac{1}{2}}\,\tilde v'^2 = \text{const.}$$

$$\tilde I(\eta)\,\tilde v'^4 = c^4$$

OPTIMALITY COND.

$$M^2 = c I^{3/2}$$
$$M = c^{1/2} I^{3/4}$$
$$I = c^{-2/3} M^{4/3}$$

$I = \pi r^4/4$; $A = \pi r^2$

$$\pi I = \left(\frac{\pi r^2}{2}\right)^2 = \left(\frac{A}{2}\right)^2$$

$$\pi I = \frac{A^2}{4}$$

$$\pi I[\xi L] = \frac{A(\xi L)^2}{4}$$

$$\pi \frac{I[\xi L]}{L^4} = \left(\frac{A(\xi L)}{L^2}\right)^2 \times \frac{1}{4}$$

$$\pi \hat I(\xi) = \frac{\hat A^2}{4}$$

$$\pi \hat I\left(\frac{\eta}{2}\right) = \frac{\hat A\left(\frac{\eta}{2}\right)^2}{4}$$

$$8\hat I\left(\frac{\eta}{2}\right) = \frac{8}{4\pi}\hat A\left(\frac{\eta}{2}\right)^2$$

$$\tilde I(\eta) = \left(\frac{2}{\pi}\right)\tilde A(\eta)^2$$

$$\int I(\eta)^{1/2}\, d\eta = V\sqrt{\frac{2}{\pi}}$$

$$\int \left(\sum \phi_p I_p\right)^{1/2} d\eta = f$$

Use in Sturm – Liouville theory:

$$L[y] = \frac{1}{w(x)}\left[-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y\right]$$

on the inner-product-space $\langle y_1, y_2\rangle = \int_a^b w y_1 y_2 \, dx$

$$\text{Rayleigh Quotient} = \frac{\langle y, Ly\rangle}{\langle y, y\rangle} = \frac{\int_a^b y(x)\left[-\frac{d}{dx}(py') + qy\right]dx}{\int_a^b wy^2 \, dx}$$

$$= \frac{-ypy'\Big|_a^b + \int_a^b (py'^2 + qy'^2)\, dx}{\int_a^b wy^2\, dx}.$$

In our case
$py' = 0$ at $x = 0$
$y = 0$ at $x = 1$
$q = 0$
$p = \tilde{I}$
$y = \tilde{v}$
$w = \tilde{F}$
$x = \eta$

$$R = \frac{\int_0^1 \tilde{I}\tilde{v}'^2 \, d\eta}{\int_0^1 \tilde{F}\tilde{v}^2 \, d\eta} \; ; \qquad \begin{array}{l} F(0) = 0 \\ F(1) = 1 \\ 0 < F(\xi) < 1 \quad \forall\, \xi \in (0,1) \end{array}$$

$$\therefore \int_0^1 \tilde{F}\tilde{v}^2 d\eta < \int_0^1 \tilde{v}^2 d\eta \Rightarrow \frac{\int_0^1 \tilde{I}\tilde{v}'^2 d\eta}{\int_0^1 \tilde{v}^2 d\eta} < \frac{\int_0^1 \tilde{I}\tilde{v}'^2 d\eta}{\int_0^1 \tilde{F}\tilde{v}^2 d\eta}$$

λ concentrated $< \lambda$ distributed:

(nice result)

Proof that the concentrated load is the most dangerous:

$$\int_0^L f(x)\,dx = F$$

$f_\xi > 0$ that all.
That is it has to be a positive distribution that's all, nothing more:

$F(1) =$ fixed:

$F(1) = 1$

$F(\xi)$.

$F(x) > F(y)$

$$\int^x f_\xi(x) \geq \int_0^y f(x)dy$$

$x > y$

$f(x) > 0$ then.

$F(x) > F(y)$

$$\int_0^L f(x)d\xi \qquad \left(\frac{\int_0^x f(x)\,d\xi}{\int_0^L f(x)\,d\xi} \right) = F(\xi)$$

P

$F(x) = 1$

$P \times F(x)$

$$\int_0^L F(\xi) v^2 d\xi \leq \frac{\int_0^1 v^2 d\xi}{1}$$

$F(\xi)$.

$F(1) = 1$ when $\xi < 1$

$F(\xi) < 1$

$$\forall v \in V . \int^1 I v'^2 d\xi$$

$$\frac{1}{\int_0^1 v^2 d\xi} \leq \frac{1}{\int_0^1 F(\xi) v^2 d\xi}$$

R_2

$\min R_2 \sim \xi = 0$

$$\int^1 F(\xi) v^2 d\xi$$

R_1

$$\frac{\int I v'^2 d\xi}{\int v^2 d\xi} \leq \frac{\int I v'^2 d\xi}{\int F(\xi) v^2 d\xi}$$

$R_1 \leq R_2$

$\min R_1 \leq \min R_2$

$\min (R_1) \leq R_2$

$\Rightarrow$ this the most critical one $\xi = 0$

Storm — L

B-C

$$F\omega^2 - \omega\int_0^x F'\omega = cA^3$$

$$2F\omega\omega' + \cancel{F'\omega^2} - \cancel{F'\omega^2} - \omega'\int_0^x F'\omega$$

$$2F\omega\omega' - \omega'\int_0^x F'\omega = \lambda A^3$$

$$\cancel{2F\omega\omega'} - \cancel{\omega'\int_0^x F'\omega}$$

$$\alpha A^2 F\omega'' + \beta F\omega = 0$$

$$\alpha A^2\omega'' + \beta\left[F\omega - \int_0^x F'\omega\right] = 0$$

$$\alpha A^2\omega'' + \beta\{F\omega - \int_0^x F'\omega\} = 0$$

$$\omega'' + \gamma A^{-2}[F\omega - \int_0^x F'\omega] = 0$$

$$\omega'' + \gamma A^{-2} \times \frac{cA^3}{\omega}$$

$$\omega'' + \gamma c\frac{A}{\omega} = 0$$

$$\omega\omega'' + \gamma cA = 0$$

$$\omega\left(F\omega - \int_0^x F'\omega\right) = cA^3$$

$$\omega'\left(F\omega - \int_0^x F'\omega\right) + \omega(F\omega' + \cancel{\omega F'} - \cancel{F'\omega}) = 3cA^2A'$$

$$\left(F\omega^2 - \omega\int_0^x F'\omega\right)^{-\frac{2}{3}} = cA^{-2}$$

$$\omega'' + \beta A^{-2}\left(F\omega - \int_0^x F'\omega\right) = 0$$

$$\omega'' + \frac{\beta\omega^{\frac{2}{3}}}{c}\left(F\omega - \int_0^x F'\omega\right)^{-\frac{2}{3}}(11) = 0$$

$$\omega'' + \lambda\omega^{-\frac{2}{3}}\left(F\omega - \int_0^x F'\omega\right)^{\frac{1}{3}} = 0$$

$$\int_{x=0}^{x} t_3\,dx = F(a)$$

$$\gamma_3 > 0$$

= Net force on

$$\int_0^{L/2} f(x)\,dx$$

$$\frac{2Pa}{2}\times\left[\frac{x}{2} - \frac{x^2}{2L}\right]$$

$$\frac{L}{2} - \frac{L^2}{4\times 2L}$$

$$\frac{2Pa}{2}\times\left(\frac{x}{2} - \frac{x^2}{2L}\right)$$

$$\left(\frac{L}{4} - \frac{L^2}{8L}\right)$$

$$\left(\frac{L}{4} - \frac{L}{8}\right)$$

$$\times\frac{L}{8}$$

$$2Pa\left(\frac{Pa}{4}\right)$$