Lower Bound Curve

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1 Extreme value of β_1

1.1 No Volume Constraint

The equation of the first order perturbation ($\mathscr{O}(\varepsilon)$) of the eigenvalue is

$$\beta_1 = \int_{\eta=0}^{1} \tilde{I}_1 \, {\tilde{v}_0}_{,\eta}^{2} \, d\eta \tag{1}$$ {eq:beta1}

where $\tilde{I}_1$ is the first order perturbation of the second moment of area $\tilde{I}_0$ and $\tilde{v}_0$ is the unperturbed eigenvector.

We want to get the maximum value of β_1 for arbitrary $\tilde{I}_1$ under the constraint that the L2 norm of the perturbation of the profile is fixed:

$$\left[\int_0^1 (r_0 - \tilde{r})^2 \, d\eta\right]^{1/2} = \varepsilon C_1 + \varepsilon^2 C_2 + \mathscr{O}(\varepsilon^3) \tag{2}$$

The L2 norm of the perturbation, if ignoring the higher order term, is $d = \varepsilon C_1$.

Let $\tilde{r} - r_0 = \varepsilon \tilde{r}_1 + \varepsilon^2 \tilde{r}_2 + \mathscr{O}(\varepsilon^3)$, then

$$\int_0^1 \left(\varepsilon^2 \tilde{r}_1^2 + 2\varepsilon^3 \tilde{r}_1 \tilde{r}_2 + \varepsilon^4 \tilde{r}_2^2 + \mathscr{O}(\varepsilon^5)\right) d\eta = \varepsilon^2 C_1^2 + 2\varepsilon^3 C_1 C_2 + \varepsilon^4 C_2^2 + \mathscr{O}(\varepsilon^5) \tag{3}$$

Which is

$$\int_0^1 \left(\tilde{r}_1^2 + 2\varepsilon \tilde{r}_1 \tilde{r}_2 + \varepsilon^2 \tilde{r}_2^2 + \mathscr{O}(\varepsilon^3)\right) d\eta = C_1^2 + 2\varepsilon C_1 C_2 + \varepsilon^2 C_2^2 + \mathscr{O}(\varepsilon^3) \tag{4}$$

If we only take the first order term ($\lim \varepsilon \to 0$), we get

$$\int_0^1 \tilde{r}_1^2 d\eta = C_1^2 \tag{5}$$

The first order relation between $\tilde{I}_1$ and $\tilde{r}_1$ is that

$$\tilde{I}_1 = \pi r_0^3 \tilde{r}_1 \tag{6}$$ {eq:I1r1}

The augmented functional of β_1 is that

$$\beta_1[\tilde{r}_1] = \int_{\eta=0}^{1} \pi r_0^3 \tilde{r}_1 \, {\tilde{v}_0}_{,\eta}^{2} \, d\eta + \lambda \left[\int_0^1 \tilde{r}_1^2 d\eta - C_1^2\right] \tag{7}$$ {eq:aug_beta1}

Do variation and let the first variation be zero

$$\delta\beta_1[\tilde{r}_1] = \int_{\eta=0}^{1} \pi r_0^3 \, \tilde{v}_{0,\eta}^{\,2} \, \delta\tilde{r}_1 d\eta + 2\lambda \int_0^1 \tilde{r}_1 \delta\tilde{r}_1 d\eta = 0 \tag{8}$$

We get the relation between λ and $\tilde{r}_1$

$$\tilde{r}_1 = -\frac{\pi r_0^3 \, \tilde{v}_{0,\eta}^{\,2}}{2\lambda} \tag{9}$$ {eq:r1}

Substitute eq. (55) to the constraint equation, we have

$$\int_0^1 \frac{\pi^2 r_0^6 \, \tilde{v}_{0,\eta}^{\,4}}{4\lambda^2} d\eta = C_1^2 \tag{10}$$

Then

$$\lambda = \frac{\pi}{2C_1} \left[\int_0^1 r_0^6 \, \tilde{v}_{0,\eta}^{\,4} \, d\eta \right]^{1/2} \tag{11}$$ {eq:lambda}

The extreme value of β_1 is that

$$\begin{aligned} \beta_1 &= \int_0^1 \pi r_0^3 \tilde{r}_1 \, \tilde{v}_{0,\eta}^{\,2} \, d\eta \\ &= -\frac{\pi^2}{2\lambda} \int_0^1 r_0^6 \, \tilde{v}_{0,\eta}^{\,4} \, d\eta \\ &= -\pi C_1 \left[\int_0^1 r_0^6 \, \tilde{v}_{0,\eta}^{\,4} \, d\eta \right]^{1/2} \end{aligned} \tag{12}$$

The relative change of the eigenvalue is a linear function of the norm of the perturbation d:

$$\begin{aligned} \frac{\tilde{\beta} - \beta_0}{\beta_0} &= \frac{\varepsilon\beta_1}{\beta_0} \\ &= -\frac{\pi\varepsilon C_1}{\beta_0} \left[\int_0^1 r_0^6 \, \tilde{v}_{0,\eta}^{\,4} \, d\eta \right]^{1/2} \\ &= -\frac{\pi}{\beta_0} \left[\int_0^1 r_0^6 \, \tilde{v}_{0,\eta}^{\,4} \, d\eta \right]^{1/2} d \end{aligned} \tag{13}$$ {eq:slope_total}

Here β_0, r_0 and $\tilde{v}_0$ are the eigenvalue, radius and eigenvector of the unperturbed column respectively.

1.2 With Volume Constraint

In the last section, we loose the constraint on the perturbation to allow the volume change, which is more practical and reasonable. However, it brings trouble in deriving the most insensitive profile of column to perturbations. Here, in addition to the constraint of fixed norm of perturbation, we also consider the constraint that the perturbation does not change the volume of the column, which means

$$\int_0^1 \pi r_0^2 d\eta = \int_0^1 \pi \left(r_0 + \varepsilon \tilde{r}_1\right)^2 d\eta = V_0 \tag{14}$$

This is essentially

$$2\varepsilon \int_0^1 r_0 \tilde{r}_1 d\eta + \varepsilon^2 \int_0^1 \tilde{r}_1^2 d\eta = 0 \tag{15}$$

Introducing an extra Lagrange multiplier λ_2 into the equation (7)

$$\beta_1[\tilde{r}_1] = \int_0^1 \pi r_0^3 \tilde{r}_1 \ \tilde{v}_{0}{}_{,\eta}^{2} \ d\eta + \lambda_1 \left[\int_0^1 \tilde{r}_1^2 d\eta - C_1^2\right] + \lambda_2 \left[\int_0^1 r_0 \tilde{r}_1 d\eta\right] \tag{16}$$ {eq:aug_beta1_CV}

Do variation and let the first variation be zero

$$\delta\beta_1[\tilde{r}_1] = \int_0^1 \pi r_0^3 \ \tilde{v}_{0}{}_{,\eta}^{2} \ \delta\tilde{r}_1 d\eta + 2\lambda_1 \int_0^1 \tilde{r}_1 \delta\tilde{r}_1 d\eta + \lambda_2 \int_0^1 r_0 \delta\tilde{r}_1 d\eta = 0 \tag{17}$$

We get the expression of $\tilde{r}_1$ in terms of λ_1 and λ_2

$$\tilde{r}_1 = -\frac{r_0\lambda_2 + \pi r_0^3 \ \tilde{v}_{0}{}_{,\eta}^{2}}{2\lambda_1} \tag{18}$$ {eq:r1_CV}

Here we can see λ_1 determines the norm of $\tilde{r}_1$. Through the norm constraint $\int_0^1 \tilde{r}_1^2 d\eta = C_1^2$ and the volume constraint $\int_0^1 \tilde{r}_0 \tilde{r}_1 d\eta$, λ_1,λ_2 can be determined as

$$\lambda_1 = \frac{\pi}{2C_1}\left[\int_0^1 r_0^6 \ \tilde{v}_{0}{}_{,\eta}^{4} \ d\eta - \frac{\pi}{V_0}\left(\int_0^1 r_0^4 \ \tilde{v}_{0}{}_{,\eta}^{2} \ d\eta\right)^2\right]^{1/2} \tag{19}$$

$$\lambda_2 = -\frac{\pi^2}{V_0}\int_0^1 r_0^4 \ \tilde{v}_{0}{}_{,\eta}^{2} \ d\eta \tag{20}$$

Plug them in equation (18), we get $\tilde{r}_1$ as

$$\tilde{r}_1 = C_1 \frac{\left(\frac{\pi}{V_0}\int_0^1 r_0^4 \ \tilde{v}_{0}{}_{,\eta}^{2} \ d\eta\right) r_0 - r_0^3 \ \tilde{v}_{0}{}_{,\eta}^{2}}{\left[\int_0^1 r_0^6 \ \tilde{v}_{0}{}_{,\eta}^{4} \ d\eta - \frac{\pi}{V_0}\left(\int_0^1 r_0^4 \ \tilde{v}_{0}{}_{,\eta}^{2} \ d\eta\right)^2\right]^{1/2}} \tag{21}$$

The extreme value of β_1 is that

$$\begin{aligned}\beta_1 &= \int_0^1 \pi r_0^3 \tilde{r}_1 \ \tilde{v}_{0}{}_{,\eta}^{2} \ d\eta \\ &= -\pi C_1 \left[\int_0^1 r_0^6 \ \tilde{v}_{0}{}_{,\eta}^{4} \ d\eta - \frac{\pi}{V_0}\left(\int_0^1 r_0^4 \ \tilde{v}_{0}{}_{,\eta}^{2} \ d\eta\right)^2\right]^{1/2}\end{aligned} \tag{22}$$

The relative change of the eigenvalue is a linear function of the norm of the perturbation d:

$$\begin{aligned}
\frac{\tilde{\beta} - \beta_0}{\beta_0} &= \frac{\varepsilon \beta_1}{\beta_0} \\
&= -\frac{\pi \varepsilon C_1}{\beta_0} \left[\int_0^1 r_0^6 \, {\tilde{v}_0}_{,\eta}^4 \, d\eta - \frac{\pi}{V_0} \left(\int_0^1 r_0^4 \, {\tilde{v}_0}_{,\eta}^2 \, d\eta \right)^2 \right]^{1/2} \qquad (23) \quad \{\texttt{eq:slope_isovolume}\} \\
&= -\frac{\pi}{\beta_0} \left[\int_0^1 r_0^6 \, {\tilde{v}_0}_{,\eta}^4 \, d\eta - \frac{\pi}{V_0} \left(\int_0^1 r_0^4 \, {\tilde{v}_0}_{,\eta}^2 \, d\eta \right)^2 \right]^{1/2} d
\end{aligned}$$

1.2.1 Application for constant cross-section half column

For constant cross-section half column of constant radius r_0

$$\beta_0 = \tilde{I}_0 \lambda_0^2 = \frac{\pi^2}{4}\tilde{I}_0, \quad \tilde{v}_{0\,,\eta} = -\frac{\pi}{\sqrt{2}} \sin\frac{\pi}{2}\eta, \quad V_0 = \pi r_0^2 \tag{24}$$

Then we can calculate the coefficient as

$$-\frac{\pi}{\beta_0}\left[\int_0^1 r_0^6\, \tilde{v}_{0\,,\eta}^{\,4}\, d\eta - \frac{\pi}{V_0}\left(\int_0^1 r_0^4\, \tilde{v}_{0\,,\eta}^{\,2}\, d\eta\right)^2\right]^{1/2} = -\frac{2\sqrt{2}}{r_0} \tag{25}$$

For $r_0 = 0.05$, the slope is -56.569, which is less steeper than the slope without volume constraint -97.9796.

1.2.2 Application for Clausen column

For the Clausen column, the profile of the half column is given in terms of a parameter θ varying from 0 to $\pi/2$:

$$r_0(\theta) = r_c \sin\theta, \tag{26}$$

$$\eta(\theta) = \frac{2}{\pi}\left(\theta - \frac{1}{2}\sin 2\theta\right) \tag{27}$$

where $r_c = \sqrt{4V_0/(3\pi L)}$. And we have

$$\beta_0 = \frac{1}{4}\frac{\pi}{3}\left(\frac{V}{L}\right)^2 = \frac{\pi}{12}V_0^2, \quad \tilde{v}_0(\theta)_{\,,\eta} = \frac{d\tilde{v}_0}{d\theta}\frac{d\theta}{d\eta} = \frac{1}{2}\pi\csc(\theta), \quad V_0 = \frac{3\pi r_c^2}{4} \tag{28}$$

Now let us rewrite the integral in parametric form

$$\begin{aligned}\int_0^1 r_0(\eta(\theta))^6\, \tilde{v}_0(\eta(\theta))_{,\eta}^{\,4}\, d\eta(\theta) &= \int_0^{\pi/2} r_0(\theta)^6\, \tilde{v}_0(\theta)_{,\eta}^{\,4}\, \eta'(\theta) d\theta \\ &= \frac{3\pi^4 r_c^6}{64}\end{aligned} \tag{29}$$

Similarly,

$$\begin{aligned}\left[\int_0^1 r_0(\eta(\theta))^4\, \tilde{v}_0(\eta(\theta))_{,\eta}^{\,2}\, d\eta(\theta)\right]^2 &= \left[\int_0^{\pi/2} r_0(\theta)^4\, \tilde{v}_0(\theta)_{,\eta}^{\,2}\, \eta'(\theta) d\theta\right]^2 \\ &= \frac{9\pi^4 r_c^8}{256}\end{aligned} \tag{30}$$

Then we can calculate the coefficient as

$$\begin{aligned}-\frac{\pi}{\beta_0}\left[\int_0^1 r_0^6\, \tilde{v}_{0\,,\eta}^{\,4}\, d\eta - \frac{\pi}{V_0}\left(\int_0^1 r_0^4\, \tilde{v}_{0\,,\eta}^{\,2}\, d\eta\right)^2\right]^{1/2} &= -\frac{\pi}{\beta_0}\left(\frac{3\pi^4 r_c^6}{64} - \frac{\pi}{V_0}\frac{9\pi^4 r_c^8}{256}\right) \\ &= 0\end{aligned} \tag{31}$$

Which means that the Clausen column is the most insensitive column to perturbation comparing to all other shape of columns.

1.3 The optimal shape for sensitivity with/without volume constraint

Recall that with volume constraint, the relative change of the eigenvalue from the worst perturbations is given by (23).

$$\frac{\tilde{\beta}-\beta_0}{\beta_0} = -\frac{\pi}{\beta_0}\left[\int_0^1 r_0^6\, \tilde{v}_0{}_{,\eta}^{4}\, d\eta - \frac{\pi}{V_0}\left(\int_0^1 r_0^4\, \tilde{v}_0{}_{,\eta}^{2}\, d\eta\right)^2\right]^{1/2} d \tag{32}$$

The slope $K_s[r_0]$ (the subscript 's' means that the perturbations only change the shape of the column) caused by isovolumetric perturbations is

$$K_s[r_0] = -\frac{\pi}{\beta_0}\left[\int_0^1 r_0^6\, \tilde{v}_0{}_{,\eta}^{4}\, d\eta - \frac{\pi}{V_0}\left(\int_0^1 r_0^4\, \tilde{v}_0{}_{,\eta}^{2}\, d\eta\right)^2\right]^{1/2} \tag{33}$$ {eq:Ks}

The slope $K_s[r_0] \leq 0$ obtains its maximum value 0 when r_0 is the profile of Clausen column. Therefore, Clausen profile is the optimal shape for imperfection sensitivity in asymptotic sense in terms of isovolumetric perturbations. However, the uniqueness is not guaranteed.

If we impose the perturbations to be $\tilde{r}_1 = -\alpha r_0$, which **keep the shape but change the volume** of the original column. The first order perturbation of the eigenvalue is

$$\beta_1[\tilde{r}_1 = -\alpha r_0] = -\int_{\eta=0}^{1} \pi\alpha r_0^4\, \tilde{v}_0{}_{,\eta}^{2}\, d\eta \tag{34}$$

The norm of the perturbations is

$$C_1 = \left(\int_0^1 \tilde{r}_1^2 d\eta\right)^{1/2} = \alpha\left(\int_0^1 r_0^2 d\eta\right)^{1/2} = \alpha\sqrt{\frac{V_0}{\pi}} \tag{35}$$

The slope $K_v[r_0]$ (the subscript 'v' means that the perturbations only change volume of the column) is

$$K_v[r_0] = -\frac{\pi}{\beta_0}\sqrt{\frac{\pi}{V_0}}\int_0^1 r_0^4\, \tilde{v}_0{}_{,\eta}^{2}\, d\eta \tag{36}$$ {eq:Kv}

Now let's look at the relative change of the eigenvalue from the worst perturbations without volume constraint given by (13).

$$\frac{\tilde{\beta}-\beta_0}{\beta_0} = -\frac{\pi}{\beta_0}\left[\int_0^1 r_0^6\, \tilde{v}_0{}_{,\eta}^{4}\, d\eta\right]^{1/2} d \tag{37}$$

The slope $K_b[r_0]$ (the subscript 'b' means that the perturbations change both the shape and the volume of the column) is

$$K_b[r_0] = -\frac{\pi}{\beta_0}\left[\int_0^1 r_0^6\, \tilde{v}_0{}_{,\eta}^{4}\, d\eta\right]^{1/2} \tag{38}$$ {eq:Kb}

The three slopes (33), (36) and (38) are related by the following equation

$$K_b[r_0]^2 = K_s[r_0]^2 + K_v[r_0]^2 \tag{39}$$

We can prove that $K_v[r_0] = -4\sqrt{\pi/V_0}$ is independent of the profile r_0 of the column. The details of the proof are described in subsection 1.3.1.

Thus the optimal profile r_0 that minimize $K_s[r_0]^2$ also minimize $K_b[r_0]^2$

$$\min_{r_0\in\mathscr{V}} K_b[r_0]^2 = \min_{r_0\in\mathscr{V}} K_s[r_0]^2 + \frac{16\pi}{V_0} \tag{40}$$

where the space of admissible functions is

$$\mathscr{V} = \left\{ r_0 \in H^2([0,1]) : \int_0^1 \pi r_0^2 = V_0 \right\}$$

Therefore, the Clausen profile is not only optimal for imperfection sensitivity with volume constraint, but also optimal for imperfection sensitivity without volume constraint.

1.3.1 Proof of $K_v[r_0]$ independent of the profile r_0 of the column

{sec:proofKv}

We are going to prove that

$$K_v[r_0] = -\frac{\pi}{\beta_0}\sqrt{\frac{\pi}{V_0}}\int_0^1 r_0^4\, \tilde{v}_0{}_{,\eta}^2\, d\eta \tag{41}$$ {eq:Kv_independent}

is independent of r_0.

Recall that the governing equation and boundary conditions for $\tilde{v}_0$,

$$(\tilde{I}_0\tilde{v}_0')' + \beta_0\tilde{v}_0 = 0, \tag{42a}$$ {eq:ZerothOrderBVP_EQ}

$$\tilde{I}_0\tilde{v}_0'\big|_{\eta=0} = 0, \tag{42b}$$ {eq:ZerothOrderBVP_BC1}

$$\tilde{v}_0\big|_{\eta=\ 1} = 0. \tag{42c}$$ {eq:ZerothOrderBVP_BC2}

Multiply (42a) by $\tilde{v}_0$ and integrate it from $\eta = 0$ to $\eta = 1$,

$$\int_0^1 (\tilde{I}_0\tilde{v}_0')'\tilde{v}_0 d\eta + \beta_0\int_0^1 \tilde{v}_0^2 d\eta = 0 \tag{43}$$

Using the normalization condition of $\tilde{v}_0$ and integrating the first term by part, we get

$$\begin{aligned}\beta_0 &= -\ \tilde{I}_0\tilde{v}_0'\tilde{v}_0\big|_{\eta=0}^{\eta=1} + \int_0^1 \tilde{I}_0\tilde{v}_0'^2 d\eta \\ &= \frac{\pi}{4}\int_0^1 r_0^4\tilde{v}_0'^2 d\eta\end{aligned} \tag{44}$$ {eq:beta0_integ}

Substituting the expression of β_0 (44) into (41), we can simplify K_v as

$$K_v[r_0] = -4\sqrt{\frac{\pi}{V_0}} \tag{45}$$

where V_0 is the constant volume of the original column.

1.4 The existence of extreme value of β_1

1.4.1 Method 1

Theorem. *Suppose that $f : C \to \mathbb{R}$, is a strongly lower semicontinuous, convex function on a strongly closed, convex, bounded subset C of a Hilbert space. Then f is bounded from below and attains its infimum.*

Consider the functional $\beta_1 : \hat{\mathscr{V}} \to \mathbb{R}$

$$\beta_1[\tilde{r}_1] = \int_{\eta=0}^{1} \pi r_0^3 \tilde{r}_1 \ \tilde{v}_{0\ ,\eta}^{\ 2} \ d\eta \tag{46}$$

where the admissible function space of $\tilde{r}_1$ is

$$\hat{\mathscr{V}} := \left\{ \tilde{r}_1 \in L^2([0,1]:R) \ : \int_0^1 \tilde{r}_1^2 d\eta \le C_1^2 \right\} \tag{47}$$ {SpaceOfAF}

This is a closed ball in a Hilbert space. **Boundedness** of the domain follows from the definition.

Now we prove the set $\hat{\mathscr{V}}$ is a **strongly closed** set. Denote $D(x_0,r) = \hat{\mathscr{V}}$ as this closed ball, where x_0 is the origin and $r = C_1$. We show that $H \setminus D$ is open. In other words, for every point $y \in H \setminus D(x_0,r)$ we need to find an open ball contained in $H \setminus D$ with center y.

Since $y \in H \setminus D(x_0,r)$, it follows that $d(y,x_0) > r$, so $d(y,x_0) - r > 0$. Let $r_1 = d(y,x_0) - r$. I claim that the open ball $B(y,r_1)$ is contained in $H \setminus D(x_0,r)$. To prove this, consider any $z \in B(y,r_1)$. Notice by the triangle inequality

$$d(x_0,y) \le d(x_0,z) + d(z,y) \Rightarrow d(z,x_0) \ge d(x_0,y) - d(z,y) > d(x_0,y) - r_1 = r \tag{48}$$

This shows $z \in H \setminus D(x_0,r)$, which completes the proof.

Now we verify that $\hat{\mathscr{V}}$ is **convex**, consider $\tilde{r} = \xi\tilde{r}_1 + (1-\xi)\tilde{r}_2$ where $\tilde{r}_1, \tilde{r}_2 \in \hat{\mathscr{V}}$ and $\xi \in [0,1]$.

$$\begin{aligned} \|\tilde{r}\| &= \|\xi\tilde{r}_1 + (1-\xi)\tilde{r}_2\| \\ &\le \|\xi\tilde{r}_1\| + \|(1-\xi)\tilde{r}_2\| \\ &\le \xi C_1 + (1-\xi)C_1 \\ &= C_1 \end{aligned} \tag{49}$$

therefore $\tilde{r} \in \hat{\mathscr{V}}$. So that $\hat{\mathscr{V}}$ is convex.

Then we verify that $\beta_1[\tilde{r}_1]$ is **strongly lower semicontinuous**. Actually, we can prove that the functional is continuous, which is a sufficient condition of strongly lower semicontinuty(A function is continuous at x_0 if and only if it is upper and lower semi-continuous there). For any real number $\varepsilon > 0$, there exists some number $\delta > 0$ such that for all $\tilde{r}_1, \tilde{r}_2 \in \hat{\mathscr{V}}$ with $\|\tilde{r}_1 - \tilde{r}_1\| < \delta$

$$|\beta_1(\tilde{r}_1) - \beta_1(\tilde{r}_2)| < \varepsilon \tag{50}$$

This comes from

$$
\begin{aligned}
|\beta_1(\tilde{r}_1)-\beta_1(\tilde{r}_2)| &= \left|\int_{\eta=0}^{1} \pi r_0^3 \tilde{r}_1\ \tilde{v}_{0}{}_{,\eta}^{2}\ d\eta - \int_{\eta=0}^{1} \pi r_0^3 \tilde{r}_2\ \tilde{v}_{0}{}_{,\eta}^{2}\ d\eta\right| \\
&= \left|\int_{0}^{1} \pi r_0^3\ \tilde{v}_{0}{}_{,\eta}^{2}\ (\tilde{r}_1-\tilde{r}_2) d\eta\right| \\
&\leq \left(\int_0^1 (\pi r_0^3\ \tilde{v}_{0}{}_{,\eta}^{2}\ )^2 d\eta \int_0^1 (\tilde{r}_1-\tilde{r}_2)^2 d\eta\right)^{1/2} \\
&= \pi\|\tilde{r}_1-\tilde{r}_1\| \left(\int_0^1 r_0^6\ \tilde{v}_{0}{}_{,\eta}^{4}\ d\eta\right)^{1/2} \\
&< \pi\delta \int_0^1 r_0^6 \left(\int_0^1 r_0^6\ \tilde{v}_{0}{}_{,\eta}^{4}\ d\eta\right)^{1/2}
\end{aligned}
\tag{51}
$$

So that we can choose $\delta = \frac{\varepsilon}{\pi\left(\int_0^1 r_0^6 \tilde{v}_{0}{}_{,\eta}^{4} d\eta\right)^{1/2}}$ to satisfy the condition of the definition. Therefore, the functional is continuous, which implies that it is strongly lower semicontinuous.

In the last, let us prove that the functional $\beta_1[\tilde{r}_1]$ is **convex**. Since this is a linear mapping, it should be convex. The proof follows

$$
\begin{aligned}
\beta_1[\xi\tilde{r}_1+(1-\xi)\tilde{r}_2] &= \int_0^1 \pi r_0^3\ \tilde{v}_{0}{}_{,\eta}^{2}\ [\xi\tilde{r}_1+(1-\xi)\tilde{r}_2] d\eta \\
&= \xi\int_0^1 \pi r_0^3\ \tilde{v}_{0}{}_{,\eta}^{2}\ \tilde{r}_1 d\eta + (1-\xi)\int_0^1 \pi r_0^3\ \tilde{v}_{0}{}_{,\eta}^{2}\ \tilde{r}_2 d\eta \\
&= \xi\beta_1[\tilde{r}_1]+(1-\xi)\beta_1[\tilde{r}_2], \quad \forall\xi\in[0,1] \quad and \quad \tilde{r}_1,\tilde{r}_2\in\hat{\mathscr{V}}
\end{aligned}
\tag{52}
$$

Therefore, the convexity is proved. And we can see from the above procedure that the functional is not strictly convex.

All in all, we have proved that all the conditions of the above theorem is satisfied. Now we can conclude that functional β_1 is bounded from below and attains its infimum.

1.4.2 Method 2

On the other hand, according to Cauchy-Schwarz inequality

$$
\left(\int_0^1 \pi r_0^3 \tilde{r}_1\ \tilde{v}_{0}{}_{,\eta}^{2}\ d\eta\right)^2 \leq \int_0^1 \left(\pi r_0^3\ \tilde{v}_{0}{}_{,\eta}^{2}\right)^2 d\eta \cdot \int_0^1 \tilde{r}_1^2 d\eta \tag{53}
$$

which gives the accessible range of $\beta_1[\tilde{r}_1]$

$$
-\pi C_1 \left[\int_0^1 r_0^6\ \tilde{v}_{0}{}_{,\eta}^{4} d\eta\right]^{1/2} \leq \beta_1[\tilde{r}_1] \leq \pi C_1 \left[\int_0^1 r_0^6\ \tilde{v}_{0}{}_{,\eta}^{4} d\eta\right]^{1/2} \tag{54}
$$

when

$$
\tilde{r}_1 = -\frac{\pi r_0^3\ \tilde{v}_{0}{}_{,\eta}^{2}}{2\lambda} \tag{55}
$$ {eq:r1}

the minimal value is obtained. Thus we proved that we got the minimal value of $\beta_1[\tilde{r}_1]$.

2 The equivalence of β_0 for full column and half column

Recall that for the full column, the governing equation is that

$$(\hat{I}\hat{w}'')'' + \beta_f(\hat{F}\hat{w}')' = 0 \quad \forall \zeta \in (0,1) \tag{56}$$ {eq:BVP_full}

$$\hat{I}\hat{w}''\big|_{\zeta=0,\ 1} = 0 \tag{57}$$

$$\hat{w}\big|_{\zeta=0,\ 1} = 0 \tag{58}$$

After following transformation

$$\eta = 2\zeta, \quad \tilde{I}(\eta) = 8\hat{I}(\eta/2), \tag{59}$$

$$\tilde{F}(\eta) = 2\hat{F}(\eta/2), \quad \tilde{w}(\eta) = \hat{w}(\eta/2), \quad \tilde{v} = \tilde{w}' \tag{60}$$

The governing equation for the half column becomes that

$$(\tilde{I}\tilde{v}')' + \beta_h\tilde{F}\tilde{v} = 0, \quad \forall \eta \in (0,1), \tag{61}$$ {eq:BVP_half}

$$\tilde{I}\tilde{v}'\big|_{\eta=0} = 0, \tag{62}$$

$$\tilde{v}\big|_{\eta=1} = 0. \tag{63}$$

To be convenient, assume that $\tilde{I}$, $\hat{I}$, $\tilde{F}$, $\hat{F}$ are independent of η or ζ. Then the solution for the full column eq. (56) is that

$$\frac{\beta_f\hat{F}}{\hat{I}} = \pi^2, \quad \hat{w} = \hat{A}\sin\pi\zeta \tag{64}$$

The solution for the half column eq. (61) is that

$$\frac{\beta_h\tilde{F}}{\tilde{I}} = \frac{\pi^2}{4}, \quad \tilde{v} = \tilde{A}\cos\frac{\pi}{2}\eta \tag{65}$$

We can see from the above results that

$$\beta_h = \frac{\pi^2\tilde{I}}{4\tilde{F}} = \frac{\pi^2 8\hat{I}}{8\hat{F}} = \frac{\pi^2\hat{I}}{\hat{F}} = \beta_f \tag{66}$$

3 Application for constant cross-section half column

3.1 The derivation of the slope

Let's consider a constant cross-section half column with dimensionless radius r_0. For the unperturbed half column, β, $\tilde{I}_0$ and $\tilde{v}$ satisfy the following equation

$$(\tilde{I}_0\tilde{v}')' + \beta\tilde{F}\tilde{v} = 0, \quad \forall\eta \in (0,1), \tag{67}$$

$$\tilde{I}_0\tilde{v}'\big|_{\eta=0} = 0, \tag{68}$$

$$\tilde{v}\big|_{\eta=1} = 0. \tag{69}$$

The general solution is that

$$\tilde{v} = c_1 \sin\lambda\eta + c_2\cos\lambda\eta \tag{70}$$

From the boundary, we know that $c_1 = 0$ and $\cos\lambda = 0$, thus the eigenvalues are given by

$$\lambda_n = \sqrt{\frac{\beta_n\tilde{F}}{\tilde{I}_0}} = n\pi + \frac{\pi}{2} \tag{71}$$

For $n = 0$, note that $\tilde{F} = 1$, we get β_0 and normalized $\tilde{v}_0$:

$$\beta_0 = \tilde{I}_0\lambda_0^2 = \frac{\pi^2}{4}\tilde{I}_0, \quad \tilde{v}_0 = \sqrt{2}\cos\frac{\pi}{2}\eta \tag{72}$$

Then we can calculate the coefficient as

$$\begin{aligned} -\frac{\pi}{\beta_0}\left[\int_0^1 r_0^6\,\tilde{v}_{0,\eta}^{\;4}\,d\eta\right]^{1/2} &= -\frac{\pi r_0^3}{\frac{\pi^2}{4}\frac{\pi}{4}r_0^4}2\frac{\pi^2}{4}\left[\int_0^1\left(\sin\frac{\pi}{2}\eta\right)^4 d\eta\right]^{1/2} \\ &= -\frac{2\sqrt{6}}{r_0} \\ &\approx -\frac{4.89898}{r_0} \end{aligned} \tag{73}$$

We consider $r_0 = 0.05$, thus the slope is -97.9796.

Note: According to our previous transformation $\tilde{I}(\eta) = 8\hat{I}(\eta/2)$ and $\hat{I} = I/L^4$, we should have

$$\hat{I} = \frac{\pi}{4}r_0^4, \quad \tilde{I} = 2\pi r_0^4 \tag{74}$$

In the above derivation, we use $\tilde{I} = \frac{\pi}{4}r_0^4$, which is not precise. If we use $\tilde{I} = 2\pi r_0^4$, the relative change of the eigenvalue becomes

$$\frac{\tilde{\beta} - \beta_0}{\beta_0} = -\frac{8\pi}{\beta_0}\left[\int_0^1 r_0^6\,\tilde{v}_{0,\eta}^{\;4}\,d\eta\right]^{1/2} d \tag{75}$$

Put $\beta_0 = \frac{\pi^2}{4}\tilde{I}_0 = 8\frac{\pi^2}{4}\frac{\pi}{4}r_0^4$ and $\tilde{v}_0 = \sqrt{2}\cos\frac{\pi}{2}\eta$ into it, we get the same result

$$-\frac{8\pi}{\beta_0}\left[\int_0^1 r_0^6\,\tilde{v}_{0,\eta}^{\;4}\,d\eta\right]^{1/2} = -\frac{2\sqrt{6}}{r_0} \approx -\frac{4.89898}{r_0} \tag{76}$$

3.2 Comparison with the numerical experiment

The above result matches well with the numerical data

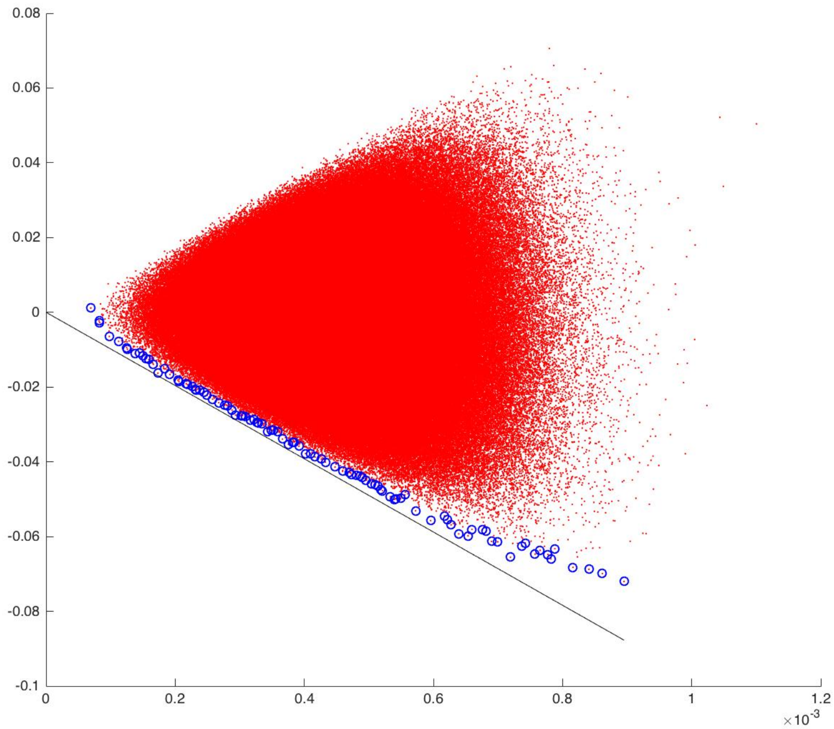

Figure 1: The lower bound of the constant cross-section half column

3.3 The relationship between $\tilde{v}_1$ and $\tilde{v}_0$

Substituting the perturbation expansions for $\tilde{I}$, $\tilde{v}$, and β in the EL equations, differentiating w.r.t ε and then setting $\varepsilon = 0$ we get that $\tilde{v}_1$ has to satisfy the following boundary value problem, {eq:FirstOrderBVP}

$$(\tilde{I}_0\tilde{v}_1')' + \beta_0\tilde{v}_1 = -(\tilde{I}_1\tilde{v}_0')' - \beta_1\tilde{v}_0 \tag{77a}$$ {eq:FirstOrderBVP_DE}

$$I_0 v_1'\big|_{\eta=0} = 0, \tag{77b}$$ {eq:FirstOrderBVP_BC1}

$$\tilde{v}_1\big|_{\eta=1} = 0. \tag{77c}$$ {eq:FirstOrderBVP_BC2}

3.3.1 For constant cross-section column

Now that we know for constant cross-section column

$$\tilde{I}_0 = \frac{\pi}{4}r_0^4, \quad \tilde{I}_1 = \pi r_0^3\tilde{r}_1, \quad \beta_0 = \frac{\pi^2}{4}\tilde{I}_0, \quad \beta_1 = -\pi C_1\left[\int_0^1 r_0^6\, \tilde{v}_{0}{}_{,\eta}^{4}\, d\eta\right]^{1/2} \tag{78}$$

$$\lambda = \frac{\pi}{2C_1}\left[\int_0^1 r_0^6\, \tilde{v}_{0}{}_{,\eta}^{4}\, d\eta\right]^{1/2}, \quad \tilde{r}_1 = -\frac{\pi r_0^3\, \tilde{v}_{0}{}_{,\eta}^{2}}{2\lambda}, \quad \tilde{v}_0 = \sqrt{2}\cos\frac{\pi}{2}\eta \tag{79}$$

Plugging all the known terms into the equation. It gives that the right hand side as

$$\begin{aligned} -(\tilde{I}_1\tilde{v}_0')' - \beta_1\tilde{v}_0 &= -\sqrt{3}C_1\pi^3 r_0^3\sin^2\frac{\pi}{2}\eta\cos\frac{\pi}{2}\eta + \pi^3 r_0^3 C_1\frac{\sqrt{3}}{4}\cos\frac{\pi}{2}\eta \\ &= \frac{\sqrt{3}}{4}\pi^3 r_0^3 C_1\cos\frac{3\pi}{2}\eta \end{aligned} \tag{80}$$

which is orthogonal to $\tilde{v}_0$, since

$$\int_0^1 \left(\frac{\sqrt{3}}{4}\pi^3 r_0^3 C_1\cos\frac{3\pi}{2}\eta\right)\left(\sqrt{2}\cos\frac{\pi}{2}\eta\right)d\eta = 0 \tag{81}$$

The governing equation becomes

$$\tilde{v}_1'' + \frac{\pi^2}{4}\tilde{v}_1 = \frac{\sqrt{3}\pi^2 C_1}{r_0}\cos\frac{3\pi}{2}\eta \tag{82}$$

which can be easily solved as

$$\tilde{v}_1 = -\frac{\sqrt{3}C_1}{2r_0}\cos\frac{3\pi}{2}\eta \tag{83}$$

It is obvious that $\tilde{v}_1$ is orthogonal to $\tilde{v}_0$.

3.3.2 For arbitrary cross-section column

If we consider other column shapes where $\tilde{I}_0$ is not a constant. We could still expect that $\tilde{v}_1$ is orthogonal to $\tilde{v}_0$.

The solution $\tilde{v}_1$ of the non-homogeneous problem (77) can be expressed as a series of the form

$$\tilde{v}_1(\eta) = \phi(\eta) = \sum_{n=1}^{\infty} b_n \phi_n(\eta) \tag{84}$$

where $\phi_1, \phi_2, \ldots, \phi_n, \ldots$ are the corresponding eigenfunctions of the homogeneous problem

$$(\tilde{I}_0 \tilde{v}')' + \lambda \tilde{v} = 0 \tag{85}$$

The fundamental eigenfunction ϕ_1 actually is just $\tilde{v}_0$ and the fundamental eigenvalue λ_1 is β_0.

$$\phi_1 = \tilde{v}_0, \quad \lambda_1 = \beta_0 \tag{86}$$

According to Sturm-Liouville theorem, for a solution to exist for $\tilde{v}_1$, it is necessary that the right hand side $f(\eta) = -(\tilde{I}_1 \tilde{v}_0')' - \beta_1 \tilde{v}_0$ be orthogonal to $\tilde{v}_0$. Actually this is also how we get the functional of β_1. Thus $f(\eta)$ can be expanded as

$$f(\eta) = \sum_{n=1}^{\infty} c_n \phi_n(\eta) \tag{87}$$

where $c_1 = 0$ since the corresponding eigenfunction $\phi_1 = \tilde{v}_0$.

The coefficient of solution $\tilde{v}_1$ can be calculated by

$$b_n = \frac{c_n}{\lambda_n - \beta_0}, \quad n = 2, 3, 4 \ldots, \quad b_1 = 0 \tag{88}$$

It is shown that $\tilde{v}_1$ is the linear combination of all the eigenfunctions except $\tilde{v}_0$. Because all the eigenfunctions are orthogonal to each other, $\tilde{v}_1$ must be orthogonal to $\tilde{v}_0$.

4 Application for elliptic cross-section half column

For the elliptic cross-section half column, $\tilde{v}_0$ can not be obtained analytically. We use the numerical method to calculate that the slope is about -175.787.

The above result matches well with the numerical data, except that the numerical data points have a small positive intercept. The intercept may come from numerical error.

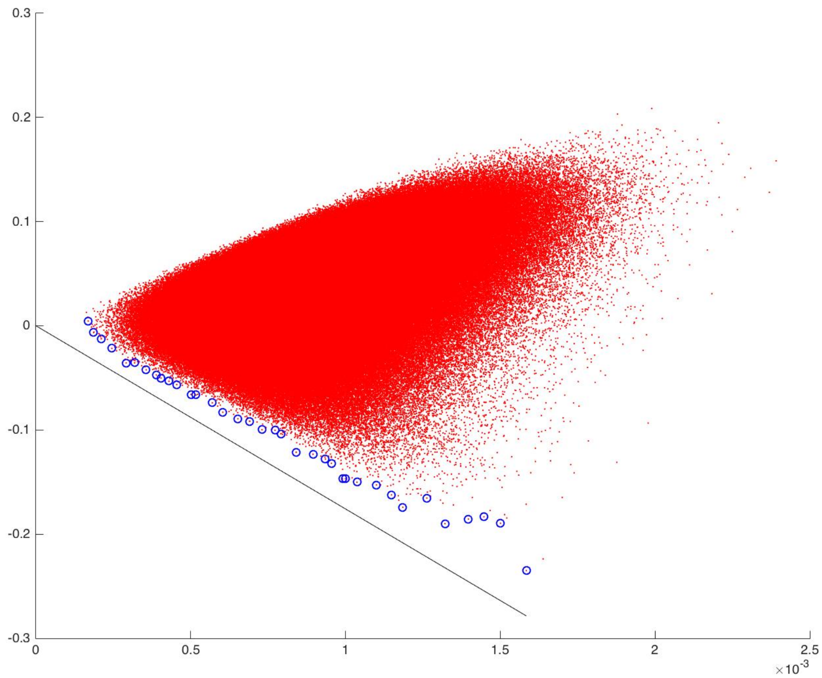

Figure 2: The lower bound of the elliptic cross-section half column

5 Application for Clausen column

For the Clausen column, the profile of the full column is given in terms of a parameter θ varying from 0 to π:

$$r_0(\theta) = r_c \sin\theta, \tag{89}$$

$$\xi(\theta) = \frac{1}{\pi}\left(\theta - \frac{1}{2}\sin 2\theta\right) \tag{90}$$

where $r_c = \sqrt{4V/(3\pi L)}$.

For the half column, let $\eta = \xi/2 \in [0,1]$, the range of $\theta \in [0,\pi/2]$.

$$r_0(\theta) = r_c \sin\theta, \tag{91}$$

$$\eta(\theta) = \frac{2}{\pi}\left(\theta - \frac{1}{2}\sin 2\theta\right) \tag{92}$$

The unperturbed eigenvalue for the half column is

$$\beta_0 = \frac{1}{4}\frac{\pi}{3}\left(\frac{V}{L}\right)^2 = \frac{\pi}{12}V^2 \tag{93}$$

The unperturbed eigenvector is $w_0(\theta) = C\sin^3\theta$.

$$\tilde{v}_0(\theta) = \frac{dw_0(\theta)}{d\eta(\theta)} = \frac{dw_0}{d\theta}\frac{d\theta}{d\eta} \tag{94}$$

After normalization, we got

$$\tilde{v}_0(\theta) = \frac{4\sin^2(\theta)\cos(\theta)}{1-\cos(2\theta)} \tag{95}$$

Then

$$\tilde{v}_0(\theta)_{,\eta} = \frac{d\tilde{v}_0}{d\theta}\frac{d\theta}{d\eta} = \frac{1}{2}\pi\csc(\theta) \tag{96}$$

Now let us rewrite the integrate in parametric form

$$\begin{aligned}\int_0^1 r_0(\eta(\theta))^6\,\tilde{v}_0(\eta(\theta))_{,\eta}^4\,d\eta(\theta) &= \int_0^{\pi/2} r_0(\theta)^6\,\tilde{v}_0(\theta)_{,\eta}^4\,\eta'(\theta)d\theta \\ &= \frac{3\pi^4 r_c^6}{64}\end{aligned} \tag{97}$$

Then we can calculate the coefficient as

$$\begin{aligned}-\frac{\pi}{\beta_0}\left[\int_0^1 r_0^6\,\tilde{v}_{0,\eta}^4\,d\eta\right]^{1/2} &= -\frac{\pi}{\frac{\pi V^2}{12}}\left[\frac{3\pi^4 r_c^6}{64}\right]^{1/2} \\ &= -4\sqrt{\frac{\pi}{V}}\end{aligned} \tag{98}$$

Here V is the volume of the column which is the same for constant, ellipse and Clausen column. $V = \pi r_{const}^2$ and we set $r_{const} = 0.05$ in this paper.

In the end, the slope of lower bound comes out to be -80 for Clausen half column. The following table summarizes the results for the three kinds of columns.

Table 1: Comparison of three kinds of columns

Cross-section	Maximum radius	Eigenvalue	Slope
Constant	0.0500	4.845E-05	-97.980
Ellipse	0.0612	5.458E-05	-175.787
Clausen	0.0577	6.467E-05	-80.000

From the above table, we can see that the Clausen column obtains the highest buckling strength. In addition, its buckling strength has the least sensitivity to perturbation compared to the other two kinds of columns.