Buckling of tapered columns: arbitrary loads and sensitivity to imperfections

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I Optimal taper for arbitrary axial loading

I.1 Variational Problem

Consider a straight beam-column of length L that in its undeformed state is orientated along the $\mathbf{e}_3$ direction. We denote the axial and transverse displacements of the columns materials points with the fields u and w. The column's ends are suppored by a rolled-type and fixed pins, respectively. Therefore, $w(0) = w(L) = 0$ and $u(L) = 0$. The colum is a composed of a homogneous materials of Young's modulus E. The column has a variable cross-section which is represented by an axially varying second moment of inertia $I(z)$. The beam-column is subject to a an axial load ditribution of the form $\mathbf{f}(z) = f(z)\mathbf{e}_3$. The total potential energy functional corresponding of the colum is I, where,

$$I[w,u] = \frac{E}{2}\int_0^L \left[A\left(u'^2 + \frac{w'^2}{2}\right)^2 + Iw''^2 \right] dz - \int_0^L fu\, dz. \tag{1}$$

For the mechanics background of this functional see Dym, and Shames p. 500. The derivation of the above potential energy functional is given in the file EBKinematics.nb.

On using the inextensibility condition

$$u' = -\frac{w'^2}{2} \tag{2}$$

the functional (1)simplifies to

$$I[w,u] = \frac{E}{2}\int_0^L Iw''^2\, dz - \int_0^L fu\, dz \tag{3}$$

Integrating the second term by parts we get

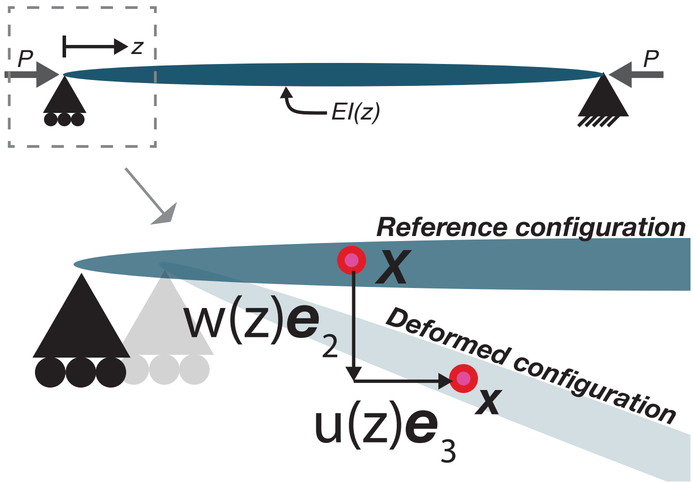

Figure 1

$$I[w,u] = \frac{E}{2}\int_0^L I w_{,z}^2\, dz - Fu|_0^L + \int_0^L F u_{,z}\, dz \tag{4}$$

where

$$F(z) := \int_0^z f(y)\, dy. \tag{5}$$

Eshelby inclusion For the specific case of far field tensile loading, modeling the spicule as an rigid inclusion and using the Eshelby theory we have that

$$f(z) = 2\frac{P_a}{L}\left(\frac{1}{2} - \frac{z}{L}\right). \tag{6}$$

The applied force $P_a = \sigma_a \pi a^2$, where σ_a is the far field applied uni-axial compression and a is the radius of the central cross-section of the beam-column. And

$$F(z) = P_a\left(\frac{z}{L} - \left(\frac{z}{L}\right)^2\right) \tag{7}$$

In terms of non-dimensional variables

$$\hat{F}(\zeta) = \zeta - \zeta^2.$$

From the essential boundary conditions of the problem $u(L) = 0$ and from the fact that $F(0) = 0$, the second term in the eq. (4) vanishes. Thus the variational problem reads,

$$\Pi[w(\cdot)] = \frac{1}{2}\int_0^L \left(EIw''^2 - Fw'^2\right)\, dz, \tag{8}$$

Note that the functional Π explicity depends only on w, and not on u. We have used the inextensibility condition (2) to simplify the third term in (4) to arrive at (8). In terms of the non-dimensional variables $\zeta = z/L$, $\beta = P_a/EL^2$, $\hat{w} = w/L$, $\hat{\Pi} = \Pi/(EL^3)$, $\hat{I} = I/L^4$, and $F/EL^2 = \beta\hat{F}$, we get

$$\hat{\Pi}[\hat{w}(\cdot)] = \frac{1}{2}\int_0^1 \left(\hat{I}\hat{w}''^2 - \beta\hat{F}\hat{w}'^2\right)\, d\zeta, \tag{9}$$

The space of admissable functions is

$$\hat{\mathcal{V}} := \left\{\hat{w} \in H^2([0,1]) \ : \ \hat{w}(0) = \hat{w}(1) = 0\right\} \tag{10}$$

I.2 Boundary Value Problem

The strong form of the problem is

$$(\hat{I}\hat{w}'')'' + \beta(\hat{F}\hat{w}')' = 0 \quad \forall\zeta \in (0,1) \tag{11}$$

$$\hat{I}\hat{w}''\big|_{\zeta=0,\ 1} = 0 \tag{12}$$

$$\hat{w}\big|_{\zeta=0,\ 1} = 0 \tag{13}$$

II Simplification of using symmetry arguments

II.1 Variational problem

We assume that the problem is completely symmetric about the point $z = L/2$. Under this assumptions the problem can be equivalently be stated as: Find

$$\tilde{v}^* = \arg\ \min\ \{\ \Pi[\tilde{v}] \ : \ \tilde{v} \in \tilde{\mathcal{V}}\ \}, \tag{14a}$$

where,

$$\tilde{\Pi}[\tilde{v}] = \int_0^1 \tilde{I}\tilde{v}'^2 - \beta\tilde{F}\tilde{v}^2\, d\eta \tag{15a}$$

$$\tilde{\mathcal{V}} := \left\{\tilde{v} \in H^1([0,1]) \ : \ \tilde{v}(1) = 0\right\} \tag{15b}$$

The function space $\tilde{\mathcal{V}}$ is called the admissible space of functions.

The relation between the new and old variables is: $\eta = 2\zeta$, $\tilde{I}(\eta) = \hat{I}(\eta/2)$, $\tilde{F}(\eta) = \hat{F}(\eta/2)$, $\tilde{w}(\eta) = \hat{w}(\eta/2)$. And $\tilde{v} = \tilde{w}'$. I do not see that there is a problem with this. If $\tilde{v}$, its anti-dervative, if t exists, will be equal to $\tilde{w}$. The constant in the anti-derivatie and be suitable chosen so as to enforce the condition that $\tilde{w}(0) = 0$. (If I recall correctly, $\tilde{\Pi}[\tilde{v}] = 2\hat{\Pi}[\hat{w}]$) I think that we should nondimensionalize β, not $\hat{F}$ so that $\tilde{F} = \hat{F}(\eta/2)$ and $\tilde{\beta} = 2\beta$. This way, $\tilde{F}$ and $\hat{F}$ both have unit magnitude and the eigenvalue is what scales.

II.2 Boundary value problem

The EL equations corresponding to the above variational problem are,

$$(\tilde{I}\tilde{v}')' + \beta\tilde{F}\tilde{v} = 0, \quad \forall\eta \in (0,1), \tag{16a}$$

$$\tilde{I}\tilde{v}'\big|_{\eta=0} = 0, \tag{16b}$$

$$\tilde{v}|_{\eta=\ 1} = 0. \tag{16c}$$

The above boundary value is special case of the Strum-Liouville problem. The problem consists of solving the second order differential equation

$$(py')' - qy + \lambda ry = 0, \quad \text{for all } \zeta \text{ in } (0,1), \tag{17a}$$

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0, \tag{17b}$$

$$\beta_1 y(1) + \beta_2 y'(1) = 0. \tag{17c}$$

Please see Ch. 11 of the book by Boyce and Di Prima for Sturm-Liouville theory.

III Optimal taper: necessary condition

III.1 Kellar's technique

Perturbation expansions Let $\tilde{I}_0$ be the optimal profile and $\tilde{v}_0$ and β_0 be the corresponding eigenfunction and eigen-values of the above Sturn-liouville problem. So, we have that

$$(\tilde{I}_0\tilde{v}_0')' + \beta_0\tilde{F}\tilde{v}_0 = 0, \quad \forall\eta \in (0,1), \tag{18a}$$

$$\tilde{I}_0\tilde{v}_0'\big|_{\eta=0} = 0, \tag{18b}$$

$$\tilde{v}_0|_{\eta=\ 1} = 0. \tag{18c}$$

Let's assume that the $\tilde{I}$, $\tilde{v}$, and β allow the following expansions,

$$\tilde{I}(\eta;\epsilon) = \tilde{I}_0(\eta) + \epsilon\tilde{I}_1(\eta) + O(\epsilon^2), \tag{19a}$$
$$\tilde{v}(\eta;\epsilon) = \tilde{v}_0(\eta) + \epsilon\tilde{v}_1(\eta) + O(\epsilon^2), \tag{19b}$$
$$\beta(\eta;\epsilon) = \beta_0 + O(\epsilon^2). \tag{19c}$$

It can be noted from the above expansions that

$$\begin{array}{ccc} \tilde{I}(\eta;\epsilon)\big|_{\epsilon=0} = \tilde{I}_0(\eta) & \tilde{I}(\eta;\epsilon)'\big|_{\epsilon=0} = \tilde{I}_0'(\eta) & \tilde{I}(\eta;\epsilon)_{,\epsilon}\big|_{\epsilon=0} = \tilde{I}_1(\eta) \\ \tilde{v}(\eta;\epsilon)|_{\epsilon=0} = \tilde{v}_0(\eta) & \tilde{v}(\eta;\epsilon)'\big|_{\epsilon=0} = \tilde{v}_0'(\eta) & \tilde{v}(\eta;\epsilon)_{,\epsilon}|_{\epsilon=0} = \tilde{v}_1(\eta) \\ \beta(\epsilon)|_{\epsilon=0} = \beta_0 & \beta(\epsilon)'\big|_{\epsilon=0} = 0 & \beta(\epsilon)_{,\epsilon}|_{\epsilon=0} = 0 \end{array}$$

Substituting the peturbation expansions for $\tilde{I}$, $\tilde{v}$, and β in the EL requations, differentiating w.r.t ϵ and then setting $\epsilon = 0$ we get that $\tilde{v}_1$ has to satisfy the following boundary value problem,

$$(\tilde{I}_0\tilde{v}_1')' + \beta_0\tilde{F}\tilde{v}_1 = -(\tilde{I}_1\tilde{v}_0')', \tag{20a}$$
$$I_0 v_1'\big|_{\eta=0} = 0, \tag{20b}$$
$$\tilde{v}_1|_{\eta=1} = 0. \tag{20c}$$

The differential equation and the second b.c. are arrived at in a straight forward manner. We discuss how we arrived at the b.c. (20b). On substituting the perturbation expansions for $\tilde{I}$, $\tilde{v}$, and β in the first b.c., (16c), taking the derivative w.r.t. ϵ, and setting $\epsilon = 0$ we get that

$$I_1 v_0' + I_0 v_1'\big|_{\eta=0} = 0. \tag{21}$$

The equation (21) in fact implies the first b.c. (20b). If $I_0(0) \neq 0$ then it follows from (18b), the b.c at $\eta = 0$ of the b.v.p relating to $(\tilde{I}_0, \tilde{v}_0, \beta_0)$ that $\tilde{v}_0'(0) = 0$. If $\tilde{v}_0'(0) = 0$ then (20b) follows from (21) . If $I_0(0) = 0$ then it again follows that $I_0(0)v_1'(0) = 0$. The b.v.p satisfied by $\tilde{v}_1$ is the non(in)-homogeneous version of the b.v.p satisfied by $\tilde{v}_0$. The in-homogeneity is the forcing term $-(\tilde{I}_1\tilde{v}_0')'$ that appears on the r.h.s of (20a) . In order for the b.v.p involving $\tilde{v}_1$ to have a solution it is necessary that the forcing term $-(\tilde{I}_1\tilde{v}_0')'$ be orthonormal to $\tilde{v}_0$. That is, it is necessary that,

$$\begin{aligned} \int_0^1 \tilde{v}_0(\tilde{I}_1\tilde{v}_0')'\,d\eta &= 0, \\ \tilde{v}_0(\tilde{I}_1\tilde{v}_0')\big|_{\eta=1} - \tilde{v}_0(\tilde{I}_1\tilde{v}_0')\big|_{\eta=0} - \int_0^1 \tilde{I}_1\tilde{v}_0'^2 &= 0, \\ \int_0^1 \tilde{I}_1\tilde{v}_0'^2 &= 0. \end{aligned} \tag{22}$$

The second equation follows by evaluting the integral in the first equation by parts. The third equation follows from the second equation by noting that the first two terms on the l.h.s of the second equation are zero . This is because it follows from (21) and (20b) that $\tilde{I}_1\tilde{v}_0'\big|_{\eta=0}=0$. Thus, the second term on the l.h.s of the second equation vanishes. And it follows from (18c) that $\tilde{v}_0|_{\eta=1}=0$, which implies that the first term on the l.h.s on the second equation also vanishes.

III.1.1 Volume constraint

All admissible profiles have to satisfy the volume constraint that

$$\sqrt{4\pi}\int_0^1 \tilde{I}^{1/2}\,d\eta=\tilde{V}, \tag{23}$$

where $L^3\tilde{V}=V$, the volume of the column. Using the perturbation expansion (19a) in (23), differentiating by ϵ, and then setting $\epsilon=0$ we get that

$$\int_0^1 \tilde{I}_0^{-1/2}\tilde{I}_1\,d\eta=0. \tag{24}$$

The optimality condition (22) is satisfied if

$$\tilde{I}_0^{1/2}\tilde{v}_0'^2=c, \tag{25}$$

where c is a real constant. This can be noted by writing the optimality condition (22) in the alternate form

$$\int_0^1 \tilde{I}_0^{-1/2}\tilde{I}_1\left(\tilde{I}_0^{1/2}\tilde{v}_0'^2\right)=0$$

and comparing it with the constraint condition (24).

III.1.2 Optimality condition

In summary the condition that characterizes the optimal profile $\tilde{I}_0$ and the corresponding eigen function $\tilde{v}_0$ is (25). Interms of the bending moment $\tilde{m}_0=\tilde{I}_0\tilde{v}_0'$, (25) reads

$$\begin{aligned}\tilde{m}_0^2&=c\tilde{I}_0^{3/2}\\ \tilde{m}_0&=c^{1/2}\tilde{I}_0^{3/4}\\ \tilde{I}_0&=c^{-2/3}m_0^{4/3}\end{aligned} \tag{26}$$

where c is a real constant. Differentiating (18a) w.r.t η, then substituting $\tilde{v}_0$ as $\tilde{m}_0/\tilde{I}_0$, and then writing $\tilde{m}_0$ interms of $\tilde{I}_0$ using the optimality condition (26), and simplifying we get that

$$\tilde{F}\tilde{I}_0\tilde{I}_0''-\tilde{F}'\tilde{I}_0'(x)\tilde{I}_0-\frac{1}{4}\tilde{F}\tilde{I}_0'^2+\frac{4}{3}\beta_0\tilde{F}^2\tilde{I}_0=0 \tag{27a}$$

$$\tilde{I}_0\big|_0=0 \tag{27b}$$

$$\tilde{I}_0'\big|_1=0 \tag{27c}$$

Given an β_0, the optimal profile $\tilde{I}_0$ can be determined by solving the above non-linear second order differential equation. The volume corresponding to the optimal profile can be determined by relating β_0 and $\tilde{V}$. The solution is propotional to $\tilde{I} \propto \tilde{V}$, in fact it appears that $\tilde{I}_0(\eta;\beta) = \beta\tilde{J}_0(\eta)$. Check this ansatz. Solve for $\tilde{J}_0$ once and fit it to a curve and you are done. The calculations are in the file Nov30.nb

III.1.3 Solutions procedure

Say $\tilde{I}_0 = \beta_0 \tilde{\mathcal{I}}_0$, where $\tilde{\mathrm{I}}_0$ is the solution of the following BVP

$$\tilde{F}\tilde{\mathcal{I}}_0\tilde{\mathcal{I}}_0'' - \tilde{F}'\tilde{\mathcal{I}}_0'(x)\tilde{\mathcal{I}}_0 - \frac{1}{4}\tilde{F}\tilde{\mathcal{I}}_0'^2 + \frac{4}{3}\tilde{F}^2\tilde{\mathcal{I}}_0 = 0 \tag{28}$$

$$\tilde{\mathcal{I}}_0(0) = 0 \tag{29}$$

$$\tilde{\mathcal{I}}_0'(1) = 0 \tag{30}$$

For a given $\tilde{F}$ the above boundary value problem can be solved. Then given an $\tilde{V}$, we can compute β_0 as

$$\beta_0 = \left(\int_0^1 \tilde{\mathcal{I}}^{1/2}\, d\eta \right)^{-2} \frac{\tilde{V}^2}{4\pi} \tag{31}$$

After computing β_0 the profile is given as $\beta_0\tilde{\mathcal{I}}_0$. The solution form the above BVP for the case $\tilde{F} = 1$ matches perfectly with that reported by Kellar. The solution of the new equation $\tilde{\mathcal{I}}_0$ matches the corresponding solution for $\tilde{I}_0$.

IV Sensitivity to Imperfections

We begin with the Rayleigh quotient characterization of the eigen value.

$$\beta_0 = \min_{\tilde{v}\in\tilde{\mathcal{V}}} \frac{\int_0^1 \tilde{I}_0 \tilde{v}_{,\zeta}^2 \, d\zeta}{\int_0^1 \tilde{F}\tilde{v}^2 \, d\zeta} \tag{32}$$

For the case of the end point loads we have that $\tilde{F}(\zeta) = 1$. The bending moments functions can be thought to be of unit norm without loss of generality. Thus, the variational problem can be written as,

$$\beta_0 = \min_{\tilde{v}\in\tilde{\mathcal{V}},\ \|\tilde{v}\|_2=1} \int_0^1 \tilde{I}_0 \tilde{v}_{,\zeta}^2 \, d\zeta \tag{33}$$

Now the question is that how does the eigen-value β_0 when you change when δI. When $\tilde{I}$ changes from $\tilde{I}_0$ to $\tilde{I}_0 + \epsilon \tilde{I}_1$, then that lead to a corresponding change in the eigen-value changes $\beta_0 + \epsilon\beta_1$, and $\tilde{v}_0 + \epsilon\tilde{v}_1$.

IV.1 Method I (will not apply to the Clausen column)

If we simply substitute the the perturbation expansions for $\tilde{I}$, β, and $\tilde{v}$ into the governing equation we get the zeroth order equation to be:

$$\tag{34}$$

similarly the first order equation is: we get that the first order equation in ϵ is

$$(\tilde{I}_0)\ \tilde{v}_{1\,,\zeta})_{,\zeta} + \beta_0 \tilde{v}_1 + (\tilde{I}_1\ \tilde{v}_{0\,,\zeta})_{,\zeta} + \beta_1 \tilde{v}_0 = 0 \tag{35}$$

$$(\tilde{I}_0)\ \tilde{v}_{1\,,\zeta})_{,\zeta} + \beta_0 \tilde{v}_1 = -(\tilde{I}_1\ \tilde{v}_{0\,,\zeta})_{,\zeta} - \beta_1 \tilde{v}_0 \tag{36}$$

The steps so far are the same that for deriving the optimal column. However, in the expansion for β we retained the first order term.

Idea I Now we really don't know at this stage as what $\tilde{I}_1$ is. However, one guess can be that $\tilde{I}_1$ that causes β_1 to assume the largest possible value is the one for which $\tilde{v}_1 = \tilde{v}_0$. However, there is no proof for this. I wold be good to get a numerical confirmation of this result before moving forward.

However, if this idea were to be pursured some of the beginning steps are as follows. If indeed $\tilde{v}_1 = \tilde{v}_0$ then the first two terms in the above equation taken together vanish. Putting the last two terms in the above equation gives us a chracterization of $\tilde{I}_1$ and β_1.

$$(\tilde{I}_1\ \tilde{v}_{0\,,\zeta})_{,\zeta} + \beta_1 \tilde{v}_0 = 0 \tag{37}$$

Multiplying both sides of the above equation with $\tilde{v}_0$ and integrating we get that

$$\beta_1 = \int_{\zeta=0}^1 \tilde{I}_1\ \tilde{v}_{0\,,\zeta}^{\ 2} \, d\zeta \tag{38}$$

In arriving at the above equation we have made use of the fact that $\|\tilde{v}_0\| = 1$.

Idea II The other alternate idea is that for a solution to exist for $\tilde{v}_1$ it is necessary that in (36) the right hand side be orthogonal to $\tilde{v}_0$. Thus multiplying both sides of (36) with $\tilde{v}_0$ and integrating we get that

IV.2 Volume constraint

However, the perturbed eigenvector is also normalized so we have that,

$$\int_0^1 \tilde{v}_0^2 \, d\zeta + 2\epsilon \int_0^1 \tilde{v}_0 \tilde{v}_1 \, d\zeta + o(\epsilon) = 1 \tag{39}$$

It appears that the normalization condition is not satisfied other than for the zeroth order. This has to be reconciled later. Perhaps it is good idea to simply ignore the volume constraint until it is actually sure that this constraint is actually necessary. Furthermore, when a spicule get's chipped it is definitely loosing volume. So, I think that it is critical to induce defects by keeping the volume constant.

A Notes and comments

1. Note that the relationship that $\tilde{I}_0 \tilde{v}'^2 = \text{const.}$ has been obtained by two different techniques. So, it should be taken seriously.
2. The maximum stress is a constant in all optimized columns.
3. We arrive at the same optimality condition by reeversing the max min and min max.
4. Can we show that $\tilde{I}_0$ should be proportinal to β_0?.–Yes
5. When is the reversing of the min max and max min allowed?—Min Max theorem by Courant.
6. Is searching for the optimal profile well posed? Read Tadabacksh's paper.

Clausen: To Do and Done

Rederived the original derivation of Keller based on the Fredlomn alternative.

Need to understand how the Rayleigh co-effcient ties into the current results.

Read Shames and Dym chapter on Buckling

Shames and Dym Chapter on Dynamics of Beams, p. 385, 389

Undersatnd the energy characterization of buckling

Derived the energy functional expressions in Shames and Dym through rigorous kinematic analysis.

I have read the introduction section of Zednec Bazant's book.

There are two types of failuer material failure and structural failure. With the advent of new materials the stability criteria have become very important because the design of structures using the new materials.

Give a quick

Proof that the solution is a global optima for arbitrary loading.

Clausen: some questions and throughs

What is the same mass ellipse corresponding to the Clausen column?

What is the imperefection sensitivity for the Clausen column?

A column embedded in an elastic matrix is equivalent to no matrix.

Is the Clausen column the global maxima?

If yes, then is it unique?

What is the complete elastica solution for the Clausen column

Need to add the result that optimizing for bening produces a cusp at the center

Why is the cross-section not triangular, if it were triangular probably the stress would not be uniform?

Spear's made of metal

Trees are subject to body weight loading, so the clausen column results would not apply to trees

The old Javelins and arrows are const. cross-sections, where as the new Javelins are tapered.

The axial filaments are sometimes triangular or square in the spicules.

Can the theory of Joseph Keller be applied to arbitrary structures (Overton)

Is the strengthening of structures agaisnt buckling well understood?

A column embedded in an elastic matrix is equivalent to no matrix.

Need to add the result that optimizing for bening produces a cusp at the center

Why the cross-section not triangular, if it were triangular probably the stress would not be uniform?

The better shape of highlighted to us by the spicule applies not just to beam-column but also to other type of frame structures. For example, a number of frame and truss type structuures are essentially assembly of columns.

The incorporation of the optimal profile column in the construction of various trusses and frame type strutures would increase the strength of those structures against buckling (structural instability failure) in a similar fashion.

For example in Figure XX. we compare the buckling strength of two different structures. The material used in both structures is the same. The nominal dimensions of both structures is also the same. However, the links in the truss structure shown in the left incorporates the clausen colums where as that in the right incorporate constant thickness links. The buckling strength increases by XX%. This proves that the stability strength enhaning due to tapeing of columns is not limited to simple columns but for other more general frame and truss type stuctures that are composed of column type links. We do not suggest that the profile given by eq. XX enahnaces the buckling strength of the struture shown in Fig. XX and YY. But only that tapering the columns will have a beneficial effect on the overall buckling strength of the strcuture.

Topology optimization has mainly been done for designing stuctures against material failure and not against structural failure. The current methodology higligted by marine structures about optimizing against buckling structures. As new materials with increased material strength are being introduced the design agaisnt buckling becomes very important.

Stability notes

Discuss the concepts of materials and strutural failure. With the increased quality if material strength the design against structural failure becomes important.

Material failure involves performing a static analysis and checking if the stresses are greater than the material strength (yield criteria, plastic limit).

The design of strcutures against strutural instability is more complicated. Here the sudden change of deformation causes the magnification of the applied forrces at certain critical points through a lever effect.

Material failure depends minimally on the geometric shape of the strucutre and more critically on the material of the strucutre and the applied forces. Wheres as structural failure depends on the geometric shape of the structures and less critically on the material and the applied forces. Since when the structural instability does occur then the applied forces are magnified through a large amount. That is to such an extent the structure would fail irrespective of what material it is composed of.